Umění vidět vědět v MAT(H)EMATICE

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Ústav informatiky AV ČR, v.v.i.

Mathematica chodila do jiné třídy

Vnitřní algoritmy, které používá *Mathematica*, mají málo společného s tím, co se učíme "běžně" ve škole.

Výsledkem je fakt, že výstup se formálně často diametrálně liší od našeho ručného výpočtu

---

Když $a + b \neq b + a$

```
b + a
```

a + b

```
HoldForm[b + a]
```

Když $a * b = \frac{c}{d}$ **pak** $a = \frac{c}{b*d}$

Vyjádření *a* z této rovnosti nemusí být tak jednoduché jako řešení z hlavy:

První reakce vede k rozčarování

```
a * b = c / d
```

```
Set::write: Tag Times in a b is Protected. >>
```

$$\frac{c}{d}$$

Pozor "=" znamená přirazení, pro "rovná se" používáme ==.

```
Solve[a * b == c / d, a]
```

$$\left\{\left\{a \to \frac{c}{b\,d}\right\}\right\}$$

Odpověd je moc zložitá

```
Solve[a * b == c/d, a]
a /. %[[1]]
```

$$\left\{\left\{a \to \frac{c}{b\,d}\right\}\right\}$$

$$\frac{c}{b\,d}$$

Menší zkrácení úprav

```
soln = Flatten[Solve[a b == c / d, a]]
expra = a /. soln
```

$$\left\{a \to \frac{c}{b\,d}\right\}$$

$$\frac{c}{b\,d}$$

Když chceme výsledek ve tvaru rovnosti:

```
sol = Solve[a b == c / d, a]
Equal @@ (sol[[1, 1]])
```

$$\left\{\left\{a \to \frac{c}{b\,d}\right\}\right\}$$

$$a == \frac{c}{b\,d}$$

Jiná varianta

```
sol /. Rule → Equal
```

$$\left\{\left\{a == \frac{c}{b\,d}\right\}\right\}$$

Hrátky pokračují

```
Reduce[a * b == c / d, a]
%[[1, 2]]
```

$$\left(b\,d \neq 0 \&\& a == \frac{c}{b\,d}\right) || (c == 0 \&\& b == 0 \&\& d \neq 0)$$

$$a == \frac{c}{b\,d}$$

Jiné řešení

```
a * b == c / d
 # / b &  /@  %
```

$$a\ b == \frac{c}{d}$$

$$a == \frac{c}{b\ d}$$

Formálně nám vadí znaménko rovnosti ve tvaru ==

```
HoldForm[a * b = c / d]
(% /. HoldForm[x_ = y_] :> Solve[x == y, x[[1]]])[[1, 1]] /.
 (x_ -> y_) -> HoldForm[x = y]
```

$$a\ b = \frac{c}{d}$$

$$a = \frac{c}{b\ d}$$

Zázračné dítě si hraje na správném pískovišti

Nezlob a počítej

Katharine

Pappas, Theoni. *Fractals, Googols and other Mathematical Tales*. Wide World Publishing / Tetra , San Carlos, Calif. 1993.

Úloha učitele Büttnera: Spočítejte čísla od 1 do 100!

Očekávané řešení :

```
soucet = 0; For[i = 1, i ≤ 10, i++, soucet = soucet + i;
 Print[soucet - i, "+", i, "=", soucet]; Pause[1]]
```

```
0+1=1
```

1+2=3

3+3=6

6+4=10

10+5=15

15+6=21

21+7=28

28+8=36

36+9=45

45+10=55

Gaußovo řešení :

```
n = 100;
m1 = Table[i, {i, 1, n}];
m2 = Table[n - i, {i, 0, n - 1}];
m3 = Table[n + 1, {i, 1, n}];
Print[TableForm[{m1, m2, m3}, TableSpacing → 1]]
```

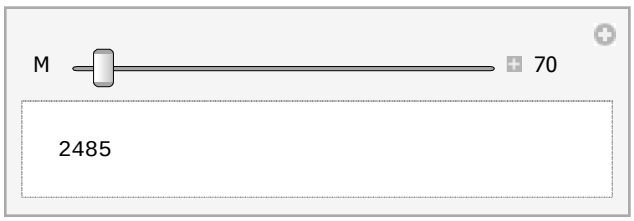
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
100	99	98	97	96	95	94	93	92	91	90	89	88	87	86	85	84	83	82	81	80
101	101	101	101	101	101	101	101	101	101	101	101	101	101	101	101	101	101	101	101	101

Celkový součet: $\frac{100 \times 101}{2} = 5050$!!!!

$$\sum_{k=1}^{100} k$$

5050

```
Manipulate[∑_{k=1}^{M} k, {{M, 100}, 1, 2000, 1, Appearance → "Labeled"}]
```



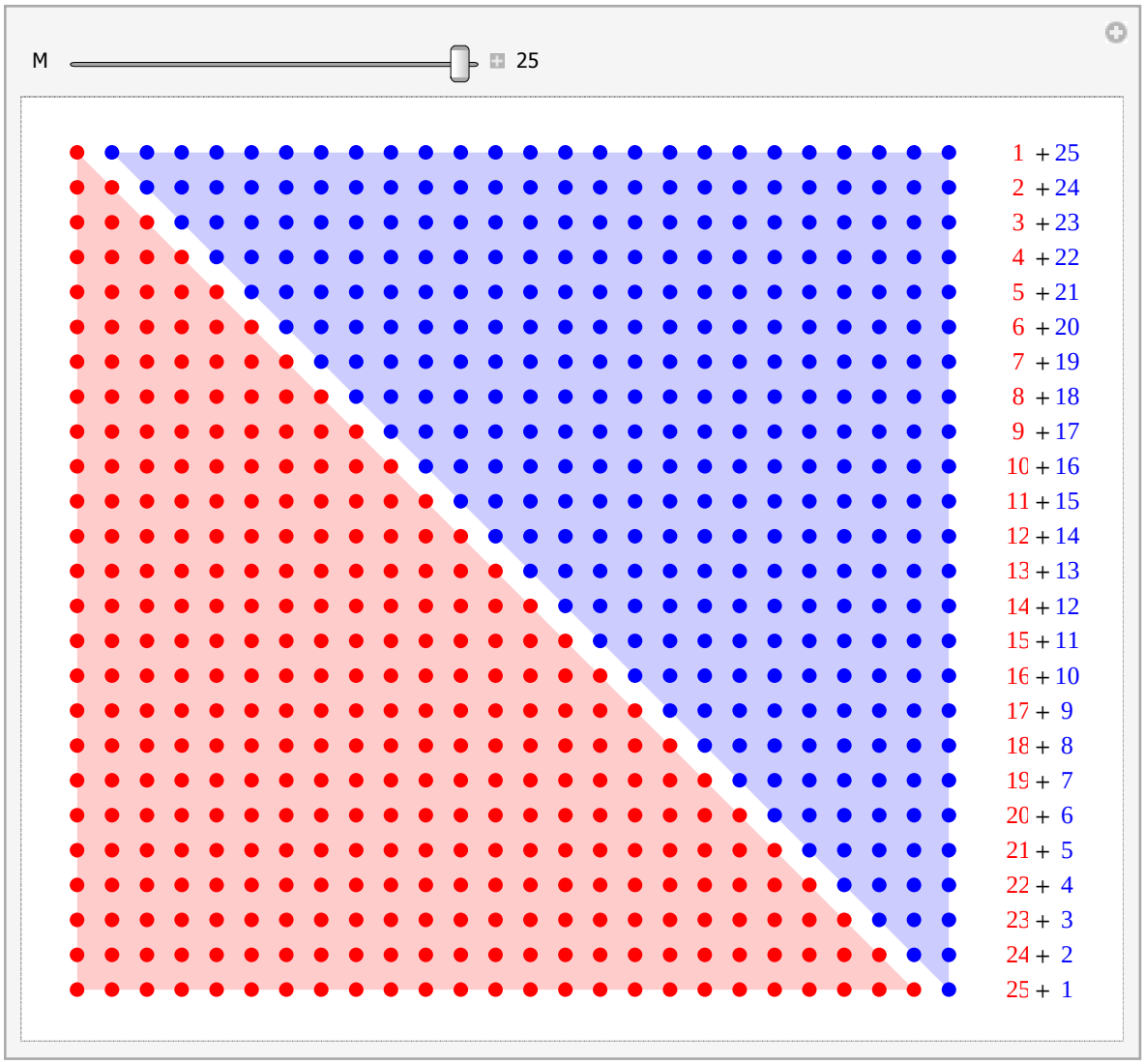
$$\sum_{k=1}^{M} k$$

$$\frac{1}{2} M (1 + M)$$

Již staří Řekové

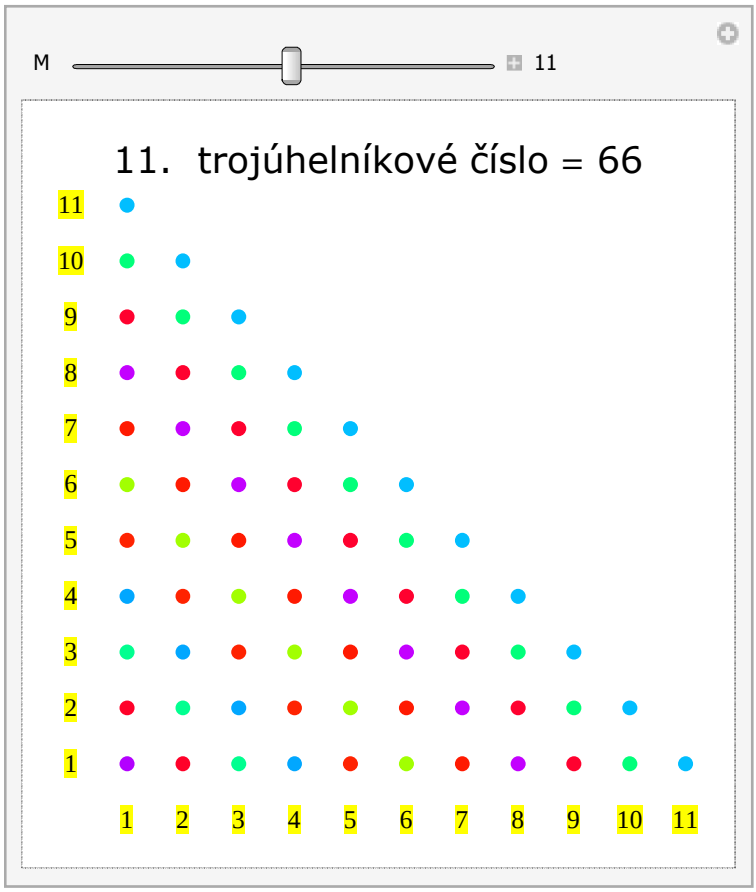
Gauß neobjevil nic nového. Již staří Řekové znali geometrickou verzi jeho důkazu:

```
Manipulate[Show[Graphics[
   {Table[{Red, Style[Text[M - j, {M + 2, j}], Medium, Background → White]}, {j, 0, M - 1}],
    Table[Style[Text[" + ", {M + 2 + .7, j}], Medium, Background → White], {j, 0, M - 1}],
    Table[{Blue, Style[Text[j + 1, {M + 2 + 1.4, j}], Medium, Background → White]},
     {j, 0, M - 1}], {Lighter[Red, .8], Polygon[{{0, 0}, {M - 1, 0}, {0, M - 1}}]},
    {Lighter[Blue, .8], Polygon[{{M, M - 1}, {M, 0}, {1, M - 1}}]},
    Table[{Red, PointSize[Large], Point[{i, j}]}, {i, 0, M - 1}, {j, 0, M - 1 - i}],
    Table[{Blue, PointSize[Large], Point[{i, j}]}, {i, 1, M}, {j, M - i, M - 1}]}],
  ImageSize → 100 * √M], {{M, 10}, 1, 25, 1, Appearance → "Labeled"}]
```



Nestačí jenom polovina obrázku?

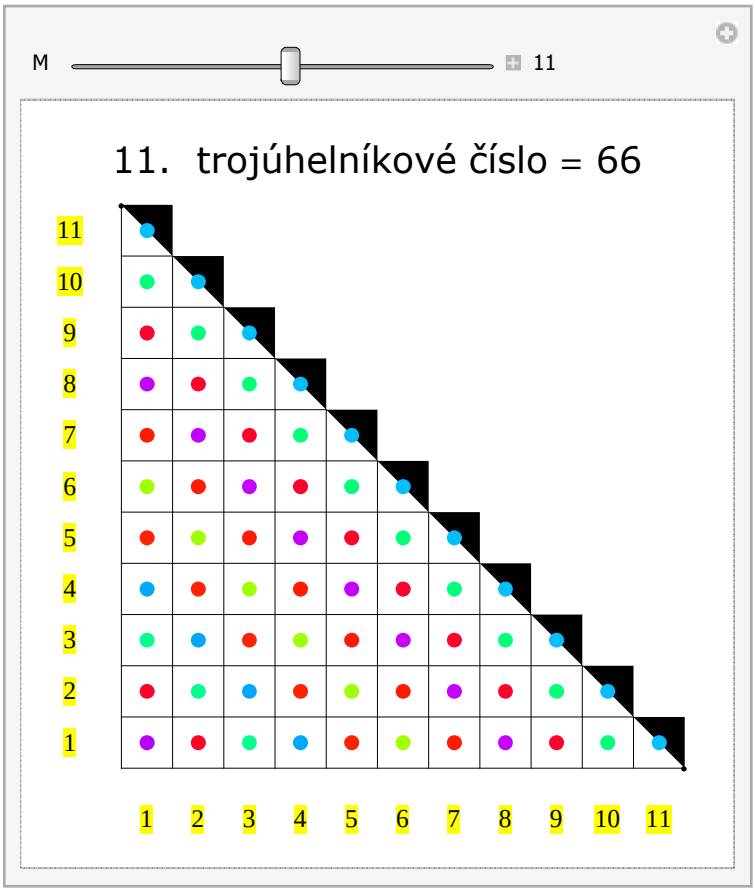
```
MM = 20; Manipulate[Graphics[
  {Table[{Black, Style[Text[j + 1, {-1, j}], Medium, Background → Yellow]}, {j, 0, M - 1}],
   Table[{Black, Style[Text[j + 1, {j, -1}], Medium, Background → Yellow]}, {j, 0, M - 1}],
   Table[{Hue[Sin[.9 * m]], PointSize[Large], Point[{i, m - 1 - i}]},
    {m, 1, M}, {i, 0, m - 1}]}, PlotLabel ->
   Style[StringJoin[ToString[M], ".  trojúhelníkové číslo = ", ToString[M (M + 1) / 2]],
    "Label", 17]  ], {{M, 11}, 1, MM, 1, Appearance → "Labeled"}]
```



V Řeckém důkazu byl použit vzorec pro plochu obdélníku, co v případě trojúhelníku

$$\sum_{k=1}^{M} k \overset{!}{=} \frac{1}{2} M M$$

```
MM = 20;
Manipulate[Graphics[{Map[Line, Table[{{0, j}, {M - j, j}}, {j, 0, M}]],
   {Black, Map[Polygon, Table[{{j, M - j}, {j + 1, M - j}, {j + 1, M - j - 1}}, {j, 0, M - 1}]]},
   Map[Line, Table[{{ j, 0}, {j, M - j}}, {j, 0, M}]],
   Table[{Black, Style[Text[j + 1, {-1, j + .5}], Medium, Background -> Yellow]},
    {j, 0, M - 1}], Table[{Black,
     Style[Text[j + 1, {j + .5, -1}], Medium, Background -> Yellow]}, {j, 0, M - 1}],
   Table[{Hue[Sin[.9 * m]], PointSize[Large], Point[{i + .5, m - 1 - i + .5}]},
    {m, 1, M}, {i, 0, m - 1}]}, PlotLabel ->
   Style[StringJoin[ToString[M], ".  trojúhelníkové číslo = ", ToString[M (M + 1) / 2]],
    "Label", 17]  ], {{M, 9}, 1, MM, 1, Appearance → "Labeled"}]
```



$$T_M = \frac{1}{2} M.M + \frac{1}{2} M = \frac{1}{2} M(M+1)$$

Trojúhelníková čísla

Trojúhelníkové číslo T_n je součtem

přirozených čísel od 1 do n, t.j.

$T_n = 1 + 2 + \ldots + n = \frac{n(n+1)}{2} = \frac{n^2+n}{2} = \binom{n+1}{2}$.

Vygenerování seznamu trojúhelníkových čísel:

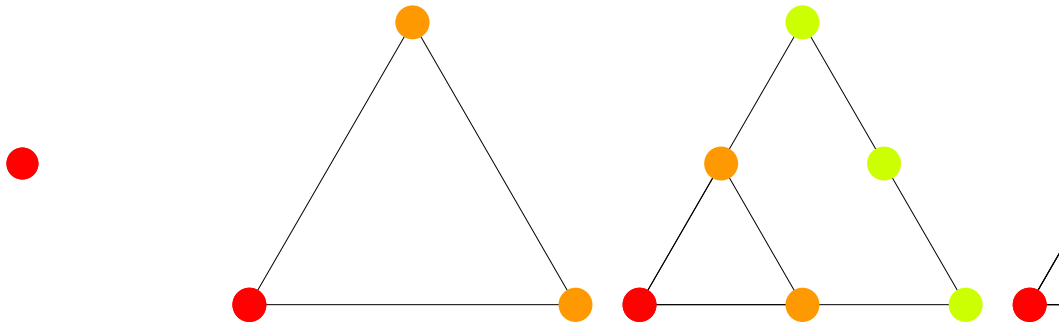
```
((#^2 + #) / 2) & /@ Range[20]
```

```
{1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210}
```

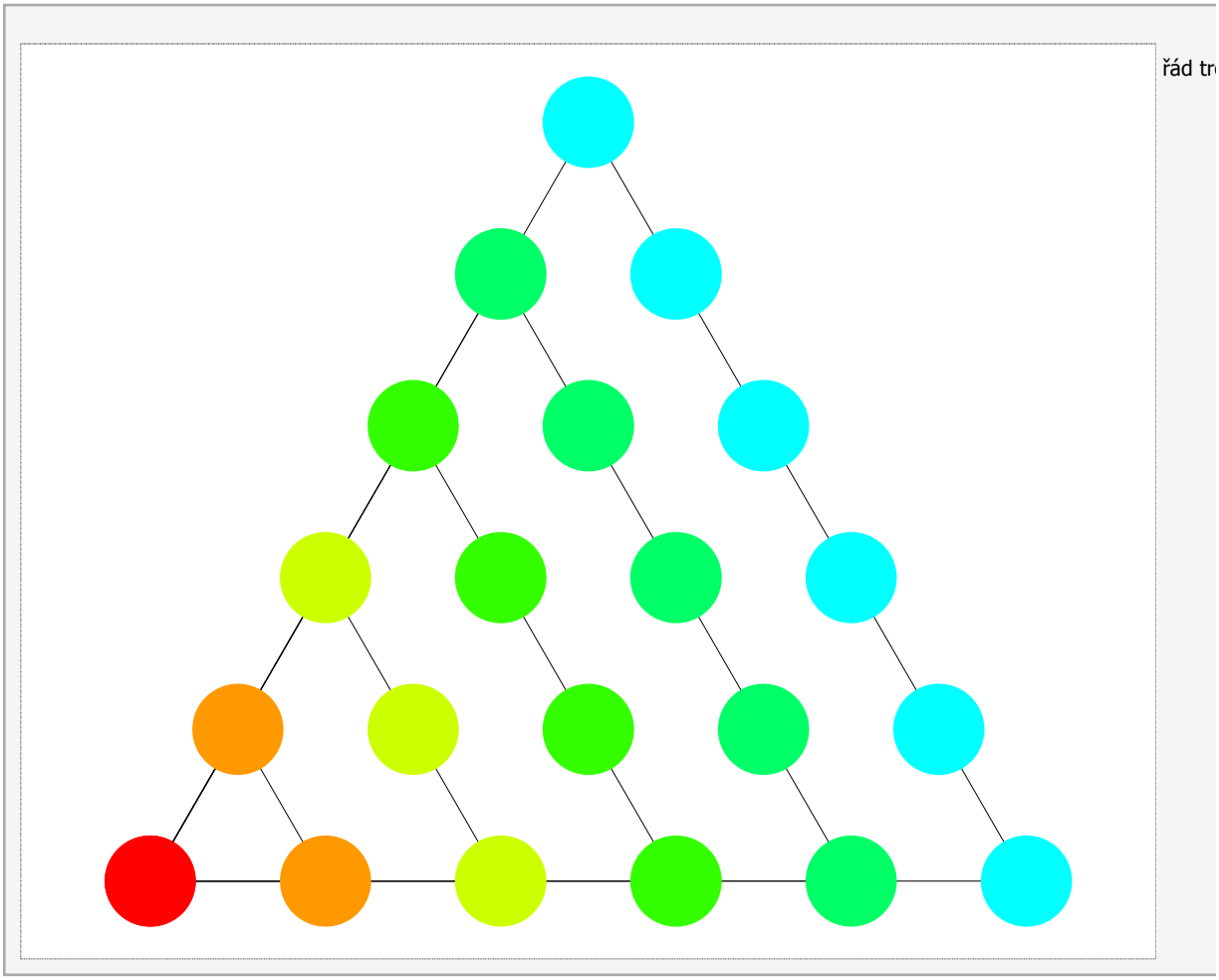
Geometricky

```
(*definicie*)
Matica[theta_] := {{Cos[theta], -Sin[theta]}, {Sin[theta], Cos[theta]}};
vertexMoj[n_, i_, m_] :=
 (m / 2) * Matica[-π (n + 2) / (2 n)].{Cos[2 Pi i / n] - 1, Sin[2 Pi i / n]} // N
maindotsMoj[n_, m_] := Table[vertexMoj[n, i, m], {i, n - 1}]
subdotsMoj[n_, m_] := Module[{i, j},
Flatten[Table[vertexMoj[n, i, m] -
     (j / m) * (vertexMoj[n, i, m] - vertexMoj[n, i - 1, m]), {j, m - 1}, {i, 2, n - 1}], 1]
]
dotsMoj[n_, m_] := {maindotsMoj[n, m], subdotsMoj[n, m]}
polygonMoj[n_, m_] := Module[{i}, Line[Table[vertexMoj[n, i, m], {i, 0, n}]]]
PolygonalDiagramMoj[n_, m_] := Module[{i}, {
   Table[polygonMoj[n, i], {i, 0, m - 1}],
   PointSize[.1],
   Table[{Hue[i / 10], Point /@ Flatten[dotsMoj[n, i], 1]}, {i, 0, m - 1}](*,
      MapIndexed[
  Text[Style[#2[[1]],14], #1] &, Union[Sequence @@ Table[Flatten[
      dots[n, i], 1], {i, 0, m - 1}]]]*)
   }
]
```

```
With[{n = 3},
 GraphicsGrid[{Graphics[#, AspectRatio → Automatic, PlotRange → All] & /@
    Table[PolygonalDiagramMoj[n, m], {m, 5}]}, ImageSize → 900]]
```



```
Manipulate[Graphics[PolygonalDiagramMoj[3, k], ImageSize → {500, 400}],
 {{k, 3, "řád trojúhelníku k"}, 1, 10, 1, Appearance → "Labeled"}, SaveDefinitions → True,
 AutorunSequencing → Automatic, ControllerLinking → True, ControlPlacement → Right]
```



Vzhledem k vyjádření pomocí kombinačního čísla, trojúhelníková čísla najdeme v třetí diagonále Pascalova trojúhelníku

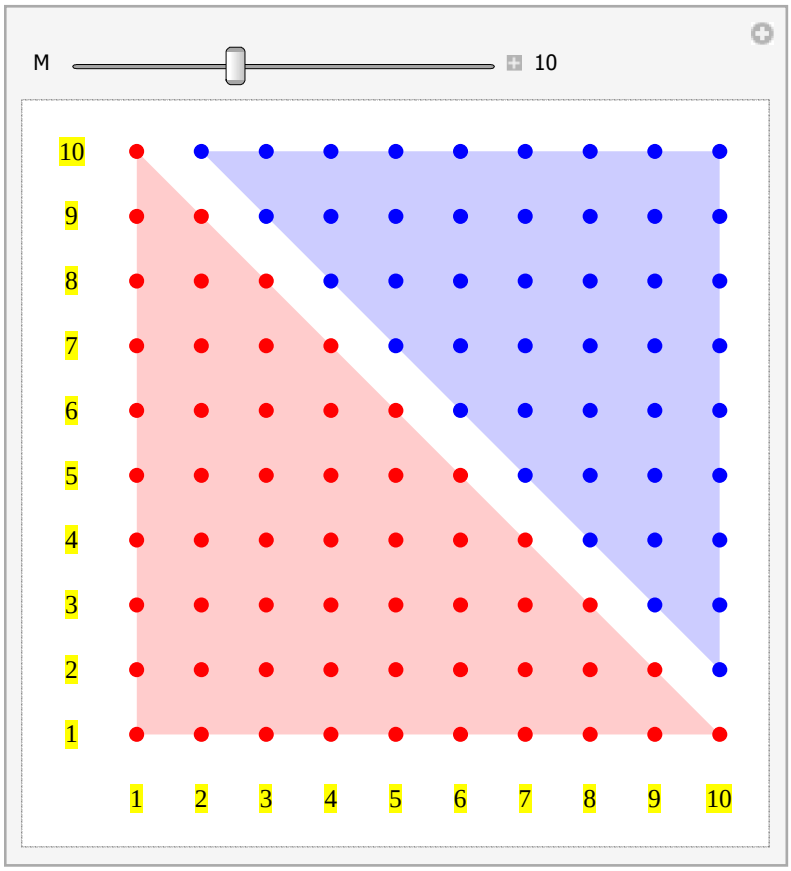
```
m = 10; ColumnForm[Table[If[(k == 2) || (k == n - 2), Style[Binomial[n, k], FontColor → Red],
   Style[Binomial[n, k], FontColor → Black]], {n, 0, m}, {k, 0, n}], Center]
```

```
                    {1}
                  {1, 1}
                {1, 2, 1}
              {1, 3, 3, 1}
            {1, 4, 6, 4, 1}
          {1, 5, 10, 10, 5, 1}
        {1, 6, 15, 20, 15, 6, 1}
      {1, 7, 21, 35, 35, 21, 7, 1}
    {1, 8, 28, 56, 70, 56, 28, 8, 1}
  {1, 9, 36, 84, 126, 126, 84, 36, 9, 1}
{1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1}
```

Theon ze Smyrny (~ 70 – ~ 135) Nikomach z Gerasy (~ (

Theon ze Smyrny a Nikomach z Gerasy věděli, že $T_n + T_{n+1} = (n+1)^2$.

```
Manipulate[Show[
  Graphics[{Table[{Black, Style[Text[j + 1, {-1, j}], Medium, Background -> Yellow]},
     {j, 0, M - 1}], Table[
     {Black, Style[Text[j + 1, {j, -1}], Medium, Background -> Yellow]}, {j, 0, M - 1}],
    {Lighter[Red, .8], Polygon[{{0, 0}, {M - 1, 0}, {0, M - 1}}]},
    {Lighter[Blue, .8], Polygon[{{M - 1, M - 1}, {M - 1, 1}, {1, M - 1}}]},
    Table[{Red, PointSize[Large], Point[{i, j}]}, {i, 0, M - 1}, {j, 0, M - 1 - i}],
    Table[{Blue, PointSize[Large], Point[{i, j}]}, {i, 1, M - 1}, {j, M - i, M - 1}]}],
  ImageSize → 100 * √M ], {{M, 10}, 1, 25, 1, Appearance → "Labeled"}]
```



Algebraický důkaz je jednoduchý, ale méně názorný

```
n =.
```

$$\sum_{k=1}^{n} k + \sum_{k=1}^{n+1} k$$

$$\frac{1}{2} n (1 + n) + \frac{1}{2} (1 + n) (2 + n)$$

```
Simplify[%]
```

$$(1 + n)^2$$

Důkaz indukcí je jasný. ↑

Výpočet druhé mocniny

Výsledek je možné použít na výpočet druhé mocniny čísel. Indentita $T_{n-1} + T_n = n^2$ po rozepsání dává

$3^2 = 1 + 1 + 2 + 2 + 3$

$4^2 = 1 + 1 + 2 + 2 + 3 + 3 + 4$

Obecně

$$n^2 = k^2 + k + (k+1) + (k+1) + \\ \ldots + (n-1) + (n-1) + n$$

Příklad:

$$253^2 = 250^2 + 250 + 251 + \\ 251 + 252 + 252 + 253 =$$

```
250^2
250 + 251 + 251 + 252 + 252 + 253
% + %%
```

62 500

1509

64 009

```
253^2
```

64 009

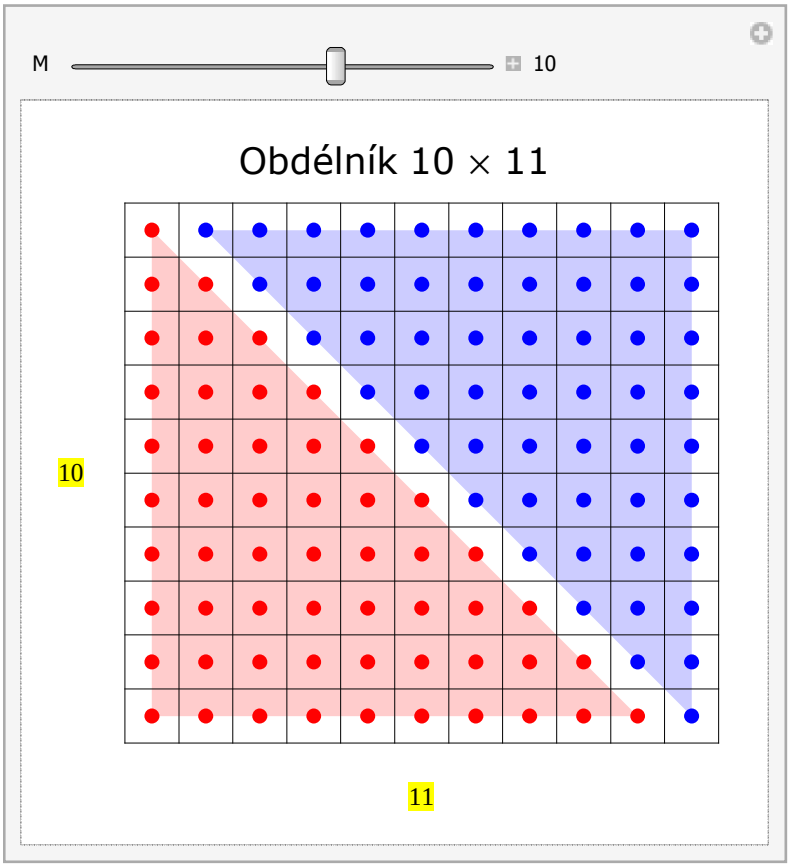
■ Test "trojúhelníkovosti" čísla

Z Gaußova důkazu plyne

$$T_n + T_n = n(n+1) \qquad \text{nebo-li}$$

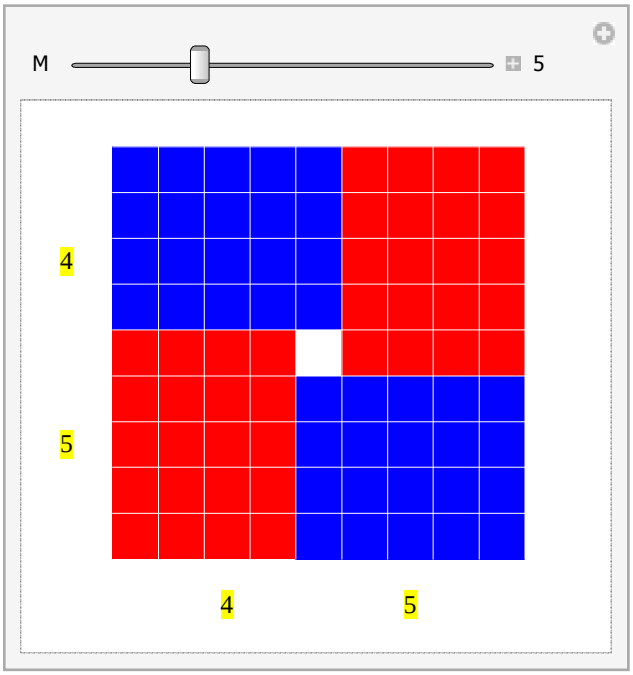
$$2\,T_n = n(n+1)$$

```
Manipulate[Show[Graphics[{
    {Lighter[Red, .8], Polygon[{{0.5, 0.5}, {M - 1 + .5, 0 + .5}, {0 + .5, M - 1 + .5}}]},
    {Lighter[Blue, .8], Polygon[{{M + .5, M - 1 + .5}, {M + .5, 0.5}, {1 + .5, M - 1 + .5}}]},
    Table[{Red, PointSize[Large], Point[{i + .5, j + .5}]}, {i, 0, M - 1}, {j, 0, M - 1 - i}],
    Table[{Blue, PointSize[Large], Point[{i + .5, j + .5}]
     }, {i, 1, M}, {j, M - i, M - 1}],
    {Black, Map[Line, Table[{{i, 0}, {i, M}}, {i, 0, M + 1}]]},
    {Black, Map[Line, Table[{{0, i}, {M + 1, i}}, {i, 0, M}]]},
    {Black, Style[Text[M, {-1, M / 2}], Medium, Background -> Yellow]},
    {Black, Style[Text[M + 1, {(M + 1) / 2, -1}], Medium, Background -> Yellow]}},
   PlotLabel -> Style[StringJoin["Obdélník ", ToString[M],
      " × ", ToString[M + 1]], "Label", 17]
  ], ImageSize → 100 * √M], {{M, 10}, 1, 15, 1, Appearance → "Labeled"}]
```

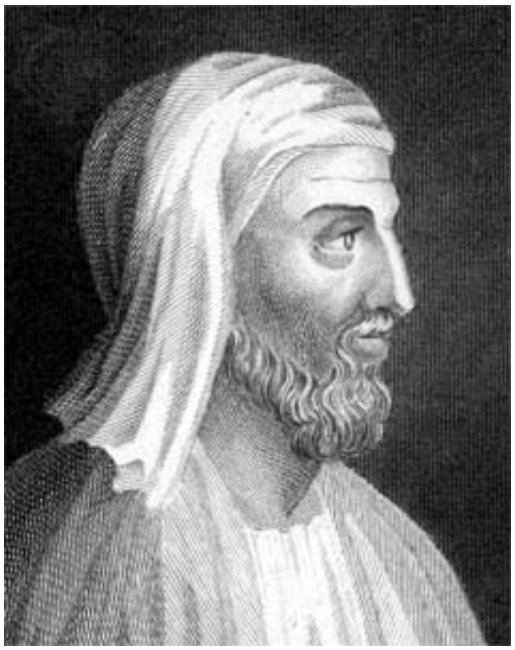


Ze čtyř takových obdélníků lehce uděláme čtverec :

```
Manipulate[Show[Graphics[{
    {Red, Polygon[{{0, 0}, {0, M}, {M - 1, M}, {M - 1, 0}}]},
    {Red, Polygon[{{M, M - 1}, {2 M - 1, M - 1}, {2 M - 1, 2 M - 1}, {M, 2 M - 1}}]},
    {Blue, Polygon[{{M - 1, 0}, {2 M - 1, 0}, {2 M - 1, M - 1}, {M - 1, M - 1}}]},
    {Blue, Polygon[{{0, M}, {M, M}, {M, 2 M - 1}, {0, 2 M - 1}}]},
    {White, Map[Line, Table[{{i, 0}, {i, 2 M - 1}}, {i, 1, 2 M - 1}]]},
    {White, Map[Line, Table[{{0, i}, {2 M - 1, i}}, {i, 1, 2 M - 1}]]},
    {Black, Style[Text[M, {-1, M / 2}], Medium, Background -> Yellow]},
    {Black, Style[Text[M - 1, {-1, M / 2 + M - 1}], Medium, Background -> Yellow]},
    {Black, Style[Text[M - 1, {M / 2, -1}], Medium, Background -> Yellow]},
    {Black, Style[Text[M, {M / 2 + M - 1, -1}], Medium, Background -> Yellow]}
  }], ImageSize → 100 * √M], {{M, 5}, 1, 15, 1, Appearance → "Labeled"}]
```



A máme vztah známý již ve starověku :

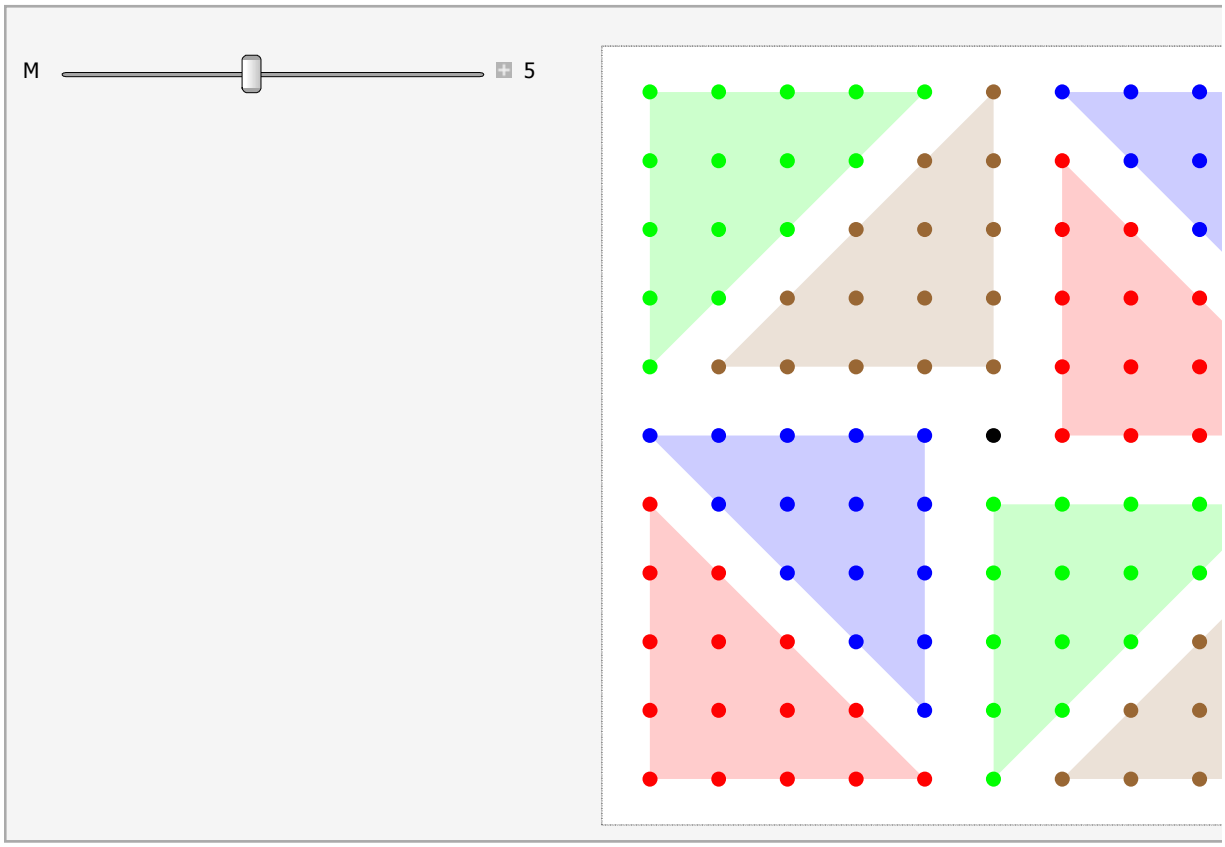


Plutarch (46 – po r .119)

Plutarch znal, že platí $8\,T_n + 1 = (2\,n + 1)^2$.

```
A[uhol_] := {{Cos[uhol], Sin[uhol]}, {-Sin[uhol], Cos[uhol]}};
obdlznik[farba1_, farba2_, index_, uhol_, posun_] := {{Lighter[farba1, .8],
   Polygon[(posun + #) & /@ ((A[uhol].# ) & /@ {{0, 0}, {index - 1, 0}, {0, index - 1}})]},
  Table[{farba1, PointSize[Large], Point[A[uhol].{i, j} + posun]},
   {i, 0, index - 1}, {j, 0, index - 1 - i}],
  {Lighter[farba2, .8], Polygon[(posun + #) & /@ ((A[uhol].# ) & /@
       {A[-π].{0, 0} + {index - 1, index}, A[-π].{index - 1, 0} + {index - 1, index},
        A[-π].{0, index - 1} + {index - 1, index}})]}, Table[{farba2, PointSize[Large],
    Point[(A[uhol].# ) & /@ {A[-π].{i, j} + {index - 1, index}} + {posun}]},
   {i, 0, index - 1}, {j, 0, index - 1 - i}]}
```

```
Manipulate[Show[Graphics[{obdlznik[Red, Blue, M, 0, {0, 0}],
    obdlznik[Red, Blue, M, 0, {M + 1, M}],
    obdlznik[Green, Brown, M, π / 2, {M, M - 1}],
    obdlznik[Green, Brown, M, π / 2, {0, 2 M}],
    {Black, PointSize[Large], Point[{M, M}]}
   }],
  ImageSize → 150 * √M], {{M, 5}, 1, 10, 2, Appearance → "Labeled"}, ControlPlacement → Left]
```

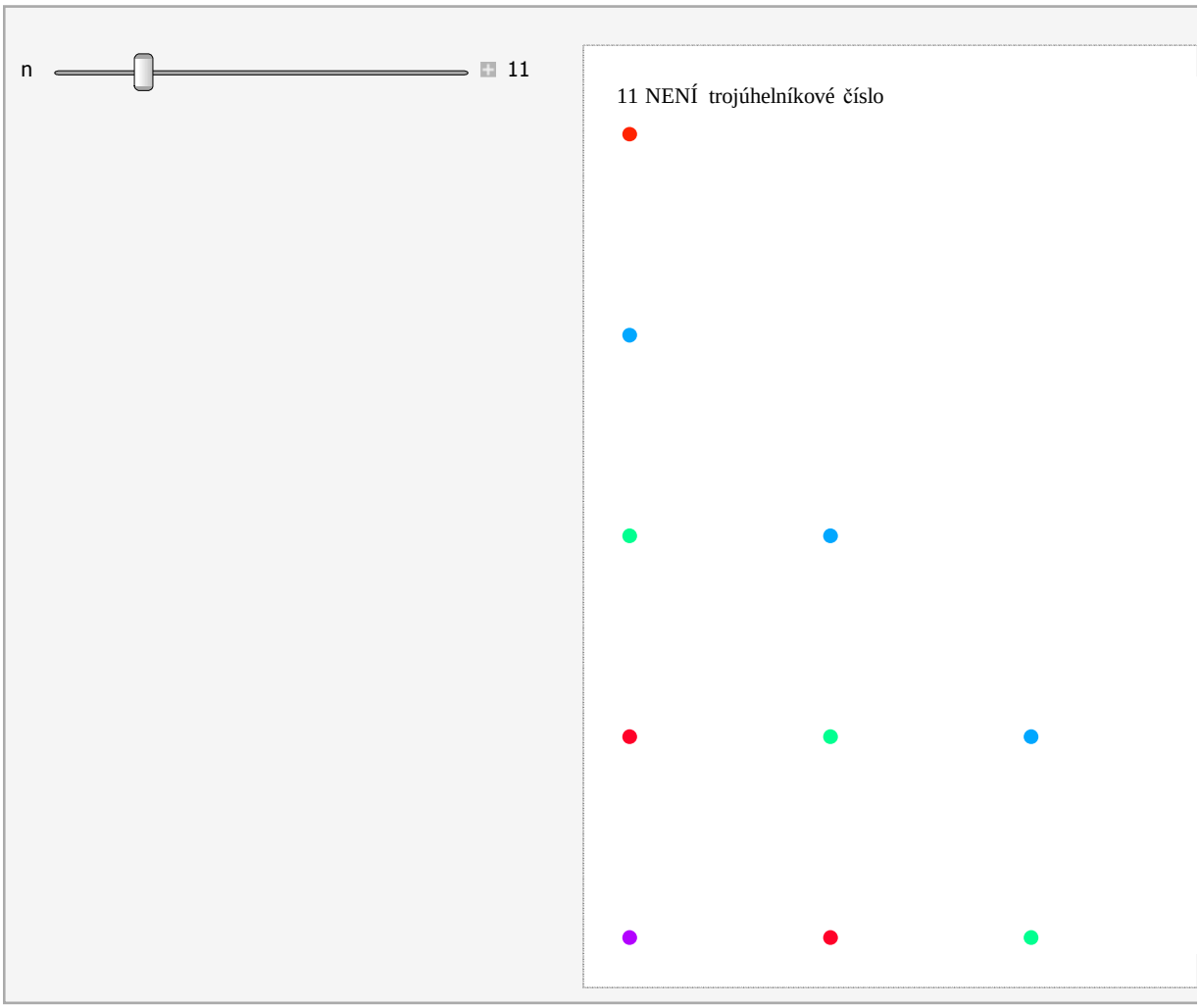


Odtud plyne jednoduchý test, zda *n* trojuhelníkové číslo :

Když $\frac{\sqrt{8n+1}-1}{2} \in \mathbb{N}$, *pak n je trojúhelníkové číslo.*

```
f[n_] := (Sqrt[8 n + 1] - 1)/2; dolneTrojuholnikove[n_] := (Floor[f[n]] (Floor[f[n]] + 1))/2;
```

```
Manipulate[Column[
  {Text@Row[{n, " ", If[IntegerQ[f[n]], " JE ", "NENÍ "], " trojúhelníkové číslo"}],
   Graphics[{Table[{Hue[Sin[.9 * m]], PointSize[Large], Point[{i, m - 1 - i}]},
      {m, 1, Floor[f[n]]}, {i, 0, m - 1}],
     Table[{Hue[Sin[.9 (Floor[f[n]] + 1)]], PointSize[Large],
       Point[{i, Ceiling[f[n]] - 1 - i}]}, {i, 0, n - dolneTrojuholnikove[n] - 1}]
    },
    ImageSize → 300]
  }],
 {{n, 11}, 3, 45, 1, Appearance → "Labeled"}, ControlPlacement → Left]
```



Algebraický důkaz je triviální (dokonce pro nutnou i postačující podmínku) :

```
k =.; n =.;
Solve[k == n (n + 1) / 2, n]
```

$$\left\{\left\{n \to \frac{1}{2}\left(-1-\sqrt{1+8\,k}\right)\right\},\ \left\{n \to \frac{1}{2}\left(-1+\sqrt{1+8\,k}\right)\right\}\right\}$$

```
n /. %
```

$$\left\{\frac{1}{2}\left(-1-\sqrt{1+8\,k}\right),\ \frac{1}{2}\left(-1+\sqrt{1+8\,k}\right)\right\}$$

Když známe podmínku můžeme generovat i jinak :

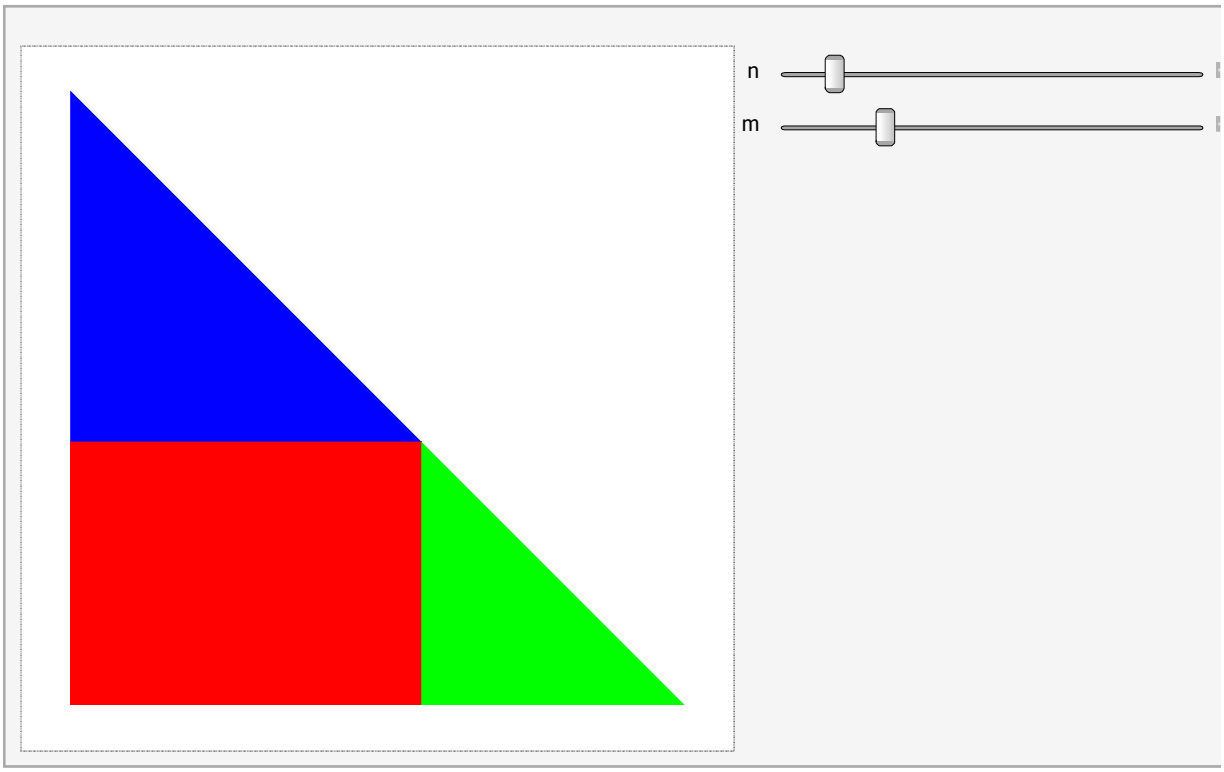
```
If[IntegerQ[f[#]], Framed[#], #] & /@ Range[45]
```

$$\{\boxed{1}, 2, \boxed{3}, 4, 5, \boxed{6}, 7, 8, 9, \boxed{10}, 11, 12, 13, 14, \boxed{15}, 16, 17, 18, 19, 20, \boxed{21}, 22, 23, 24, 25, 26, 27, \boxed{28}, 29, 30, 31, 32, 33, 34, 35, \boxed{36}, 37, 38, 39, 40, 41, 42, 43, 44, \boxed{45}\}$$

Z geometrického názoru okamžitě plyne

$$T_{n+m} = T_n + T_m + n\,m$$

```
Manipulate[Graphics[{
   {Red, Rectangle[{0, 0}, {n, m}]},
   {Blue, Polygon[{{0, m}, {n, m}, {0, m + n}}]},
   {Green, Polygon[{{n, 0}, {n + m, 0}, {n, m}}]}}],
 {{n, 4, "n"}, 3, 14, 1, Appearance → "Labeled"},
 {{m, 3, "m"}, 1, 10, 1, Appearance → "Labeled"},
 ControllerLinking → True, ControlPlacement → Right]
```




```
n = .; m = .;
```

$$\frac{1}{2} n (n+1) + \frac{1}{2} m (m+1) + n m$$

$$\frac{1}{2} m (1+m) + m n + \frac{1}{2} n (1+n)$$

```
Simplify[%]
```

$$\frac{1}{2} \left(m + m^2 + n + 2 m n + n^2\right)$$

```
Factor[%]
```

$$\frac{1}{2} (m+n) (1+m+n)$$

Multi-gonální čísla

- **Čtvercová trojúhelníková čísla**

L.Euler dne 10.srpna 1730 poznamenal, že když

```
a[n_Integer] := (3 + 2 √2)^n; b[n_Integer] := (3 - 2 √2)^n;
```

pak pro každé n je číslo $\frac{a-b}{4\sqrt{2}}$

- přirozeným číslem a
- jeho druhá mocnina je trojúhelníkové číslo. ↑

Důsledek: Existuje nekonečně mnoho trojúhelníkových čísel, která jsou čtvercem.

```
aa[n_] := (a[n] - b[n])/(4 √2)
```

```
IntegerQ[aa[n]]
```

```
False
```

```
IntegerQ[aa[1]]
IntegerQ[aa[2]]
IntegerQ[Simplify[aa[2]]]
```

```
True
```

```
False
```

```
True
```

```
Table[Simplify[aa[n]], {n, 45, 55}]
```

$$\left\{4\,980\,084\,264\,897\,221\,429\,612\,265\,939\,280\,525\,,\ 29\,026\,058\,213\,048\,656\,016\,282\,889\,493\,864\,074\,,\right.$$
$$169\,176\,265\,013\,394\,714\,668\,085\,071\,023\,903\,919\,,\ 986\,031\,531\,867\,319\,631\,992\,227\,536\,649\,559\,440\,,$$
$$5\,747\,012\,926\,190\,523\,077\,285\,280\,148\,873\,452\,721\,,\ 33\,496\,046\,025\,275\,818\,831\,719\,453\,356\,591\,156\,886\,,$$
$$195\,229\,263\,225\,464\,389\,913\,031\,439\,990\,673\,488\,595\,,\ \frac{-\left(3-2\sqrt{2}\right)^{52}+\left(3+2\sqrt{2}\right)^{52}}{4\sqrt{2}},$$
$$\frac{\left(-3+2\sqrt{2}\right)^{53}+\left(3+2\sqrt{2}\right)^{53}}{4\sqrt{2}},\ \frac{-\left(3-2\sqrt{2}\right)^{54}+\left(3+2\sqrt{2}\right)^{54}}{4\sqrt{2}},\ \left.\frac{\left(-3+2\sqrt{2}\right)^{55}+\left(3+2\sqrt{2}\right)^{55}}{4\sqrt{2}}\right\}$$

```
Table[Simplify[aa[n], ComplexityFunction -> LeafCount], {n, 45, 55}]
```

```
{4 980 084 264 897 221 429 612 265 939 280 525 ,
 29 026 058 213 048 656 016 282 889 493 864 074 , 169 176 265 013 394 714 668 085 071 023 903 919 ,
 986 031 531 867 319 631 992 227 536 649 559 440 , 5 747 012 926 190 523 077 285 280 148 873 452 721 ,
 33 496 046 025 275 818 831 719 453 356 591 156 886 , 195 229 263 225 464 389 913 031 439 990 673 488 595 ,
 1 137 879 533 327 510 520 646 469 186 587 449 774 684 ,
 6 632 047 936 739 598 733 965 783 679 534 025 159 509 ,
 38 654 408 087 110 081 883 148 232 890 616 701 182 370 ,
 225 294 400 585 920 892 564 923 613 664 166 181 934 711 }
```

Stačí změnit postup výpočtu:

```
Table[Simplify[(Expand[a[n]] - Expand[b[n]])/(4 Sqrt[2])], {n, 45, 160}]
```

```
{4 980 084 264 897 221 429 612 265 939 280 525 ,
 29 026 058 213 048 656 016 282 889 493 864 074 , 169 176 265 013 394 714 668 085 071 023 903 919 ,
 986 031 531 867 319 631 992 227 536 649 559 440 , 5 747 012 926 190 523 077 285 280 148 873 452 721 ,
 33 496 046 025 275 818 831 719 453 356 591 156 886 , 195 229 263 225 464 389 913 031 439 990 673 488 595 ,
 1 137 879 533 327 510 520 646 469 186 587 449 774 684 ,
 6 632 047 936 739 598 733 965 783 679 534 025 159 509 ,
 38 654 408 087 110 081 883 148 232 890 616 701 182 370 ,
 225 294 400 585 920 892 564 923 613 664 166 181 934 711 ,
 1 313 111 995 428 415 273 506 393 449 094 380 390 425 896 ,
 7 653 377 571 984 570 748 473 437 080 902 116 160 620 665 ,
 44 607 153 436 479 009 217 334 229 036 318 316 573 298 094 ,
 259 989 543 046 889 484 555 531 937 137 007 783 279 167 899 ,
 1 515 330 104 844 857 898 115 857 393 785 728 383 101 709 300 ,
 8 831 991 086 022 257 904 139 612 425 577 362 515 331 087 901 ,
 51 476 616 411 288 689 526 721 817 159 678 446 708 884 818 106 ,
 300 027 707 381 709 879 256 191 290 532 493 317 737 977 820 735 ,
 1 748 689 627 878 970 586 010 425 926 035 281 459 718 982 106 304 ,
 10 192 110 059 892 113 636 806 364 265 679 195 440 575 914 817 089 ,
```

59 403 970 731 473 711 234 827 759 668 039 891 183 736 506 796 230 ,
346 231 714 328 950 153 772 160 193 742 560 151 661 843 125 960 291 ,
2 017 986 315 242 227 211 398 133 402 787 321 018 787 322 248 965 516 ,
11 761 686 177 124 413 114 616 640 222 981 365 961 062 090 367 832 805 ,
68 552 130 747 504 251 476 301 707 935 100 874 747 585 219 958 031 314 ,
399 551 098 307 901 095 743 193 607 387 623 882 524 449 229 380 355 079 ,
2 328 754 459 099 902 322 982 859 936 390 642 420 399 110 156 324 099 160 ,
13 572 975 656 291 512 842 153 966 010 956 230 639 870 211 708 564 239 881 ,
79 109 099 478 649 174 729 940 936 129 346 741 418 822 160 095 061 340 126 ,
461 081 621 215 603 535 537 491 650 765 124 217 873 062 748 861 803 800 875 ,
2 687 380 627 814 972 038 495 008 968 461 398 565 819 554 333 075 761 465 124 ,
15 663 202 145 674 228 695 432 562 160 003 267 177 044 263 249 592 764 989 869 ,
91 291 832 246 230 400 134 100 363 991 558 204 496 446 025 164 480 828 474 090 ,
532 087 791 331 708 172 109 169 621 789 345 959 801 631 887 737 292 205 854 671 ,
3 101 234 915 744 018 632 520 917 366 744 517 554 313 345 301 259 272 406 653 936 ,
18 075 321 703 132 403 623 016 334 578 677 759 366 078 439 919 818 342 234 068 945 ,
105 350 695 303 050 403 105 577 090 105 322 038 642 157 294 217 650 780 997 759 734 ,
614 028 850 115 170 015 010 446 206 053 254 472 486 865 325 386 086 343 752 489 459 ,
3 578 822 405 387 969 686 957 100 146 214 204 796 279 034 658 098 867 281 517 177 020 ,
20 858 905 582 212 648 106 732 154 671 231 974 305 187 342 623 207 117 345 350 572 661 ,
121 574 611 087 887 918 953 435 827 881 177 641 034 845 021 081 143 836 790 586 258 946 ,
708 588 760 945 114 865 613 882 812 615 833 871 903 882 783 863 655 903 398 166 981 015 ,
4 129 957 954 582 801 274 729 861 047 813 825 590 388 451 682 100 791 583 598 415 627 144 ,
24 071 158 966 551 692 782 765 283 474 267 119 670 426 827 308 741 093 598 192 326 781 849 ,
140 296 995 844 727 355 421 861 839 797 788 892 432 172 512 170 345 770 005 555 545 063 950 ,
817 710 816 101 812 439 748 405 755 312 466 234 922 608 245 713 333 526 435 140 943 601 851 ,
4 765 967 900 766 147 283 068 572 692 077 008 517 103 476 962 109 655 388 605 290 116 547 156 ,
27 778 096 588 495 071 258 663 030 397 149 584 867 698 253 526 944 598 805 196 599 755 681 085 ,
161 902 611 630 204 280 268 909 609 690 820 500 689 086 044 199 557 937 442 574 308 417 539 354 ,
943 637 573 192 730 610 354 794 627 747 773 419 266 818 011 670 403 025 850 249 250 749 555 039 ,
5 499 922 827 526 179 381 859 858 156 795 820 014 911 822 025 822 860 217 658 921 196 079 790 880 ,
32 055 899 391 964 345 680 804 354 313 027 146 670 204 114 143 266 758 280 103 277 925 729 190 241 ,
186 835 473 524 259 894 702 966 267 721 367 060 006 312 862 833 777 689 462 960 746 358 295 350 566 ,
1 088 956 941 753 595 022 536 993 252 015 175 213 367 673 062 859 399 378 497 661 200 224 042 913 155 ,
6 346 906 176 997 310 240 518 993 244 369 684 220 199 725 514 322 618 581 523 006 454 985 962 128 364 ,
36 992 480 120 230 266 420 576 966 214 202 930 107 830 680 023 076 312 110 640 377 529 691 729 857 029 ,
215 607 974 544 384 288 282 942 804 040 847 896 426 784 354 624 135 254 082 319 258 723 164 417 013 810 ,
1 256 655 367 146 075 463 277 079 858 030 884 448 452 875 447 721 735 212 383 275 174 809 294 772 225 831
,
7 324 324 228 332 068 491 379 536 344 144 458 794 290 468 331 706 276 020 217 331 790 132 604 216 341 176
,
42 689 290 002 846 335 485 000 138 206 835 868 317 289 934 542 515 920 908 920 715 565 986 330 525 821 ↘
225,
248 811 415 788 745 944 418 621 292 896 870 751 109 449 138 923 389 249 433 306 961 605 785 378 938 586 ↘
174,
1 450 179 204 729 629 331 026 727 619 174 388 638 339 404 898 997 819 575 690 921 054 068 725 943 105 ↘
695 819,
8 452 263 812 589 030 041 741 744 422 149 461 078 926 980 255 063 528 204 712 219 362 806 570 279 695 ↘
588 740,
49 263 403 670 804 550 919 423 738 913 722 377 835 222 476 631 383 349 652 582 395 122 770 695 735 067 ↘
836 621,
287 128 158 212 238 275 474 800 689 060 184 805 932 407 879 533 236 569 710 782 151 373 817 604 130 711 ↘
430 986,
1 673 505 545 602 625 101 929 380 395 447 386 457 759 224 800 568 036 068 612 110 513 120 134 929 049 ↘
200 749 295,
9 753 905 115 403 512 336 101 481 683 624 133 940 622 940 923 874 979 841 961 880 927 346 991 970 164 ↘
493 064 784,
56 849 925 146 818 448 914 679 509 706 297 417 185 978 420 742 681 842 983 159 175 050 961 816 891 937 ↘
757 639 409,
331 345 645 765 507 181 151 975 576 554 160 369 175 247 583 532 216 078 056 993 169 378 423 909 381 462 ↘
052 771 670,

```
1 931 223 949 446 224 637 997 173 949 618 664 797 865 507 080 450 614 625 358 799 841 219 581 639 396 ↘
  834 558 990 611 ,
11 255 998 050 911 840 646 831 068 121 157 828 418 017 794 899 171 471 674 095 805 877 939 065 926 999 ↘
  545 301 171 996 ,
65 604 764 356 024 819 242 989 234 777 328 305 710 241 262 314 578 215 419 216 035 426 414 813 922 600 ↘
  437 248 041 365 ,
382 372 588 085 237 074 811 104 340 542 812 005 843 429 778 988 297 820 841 200 406 680 549 817 608 603 ↘
  078 187 076 194 ,
2 228 630 764 155 397 629 623 636 808 479 543 729 350 337 411 615 208 709 627 986 404 656 884 091 729 ↘
  018 031 874 415 799 ,
12 989 411 996 847 148 702 930 716 510 334 450 370 258 594 690 702 954 436 926 718 021 260 754 732 765 ↘
  505 113 059 418 600 ,
75 707 841 216 927 494 587 960 662 253 527 158 492 201 230 732 602 517 911 932 321 722 907 644 304 864 ↘
  012 646 482 095 801 ,
441 257 635 304 717 818 824 833 257 010 828 500 582 948 789 704 912 153 034 667 212 316 185 111 096 418 ↘
  570 765 833 156 206 ,
2 571 837 970 611 379 418 361 038 879 811 443 845 005 491 507 496 870 400 296 070 952 174 203 022 273 ↘
  647 411 948 516 841 435 ,
14 989 770 188 363 558 691 341 400 021 857 834 569 450 000 255 276 310 248 741 758 500 729 033 022 545 ↘
  465 900 925 267 892 404 ,
87 366 783 159 569 972 729 687 361 251 335 563 571 694 510 024 160 991 092 154 480 052 199 995 112 999 ↘
  147 993 603 090 512 989 ,
509 210 928 769 056 277 686 782 767 486 155 546 860 717 059 889 689 636 304 185 121 812 470 937 655 449 ↘
  422 060 693 275 185 530 ,
2 967 898 789 454 767 693 391 009 243 665 597 717 592 607 849 313 976 826 732 956 250 822 625 630 819 ↘
  697 384 370 556 560 600 191 ,
17 298 181 807 959 549 882 659 272 694 507 430 758 694 930 035 994 171 324 093 552 383 123 282 847 262 ↘
  734 884 162 646 088 415 616 ,
100 821 192 058 302 531 602 564 626 923 378 986 834 576 972 366 651 051 117 828 358 047 917 071 452 756 ↘
  711 920 605 319 969 893 505 ,
587 628 970 541 855 639 732 728 488 845 766 490 248 766 904 163 912 135 382 876 595 904 379 145 869 277 ↘
  536 639 469 273 730 945 414 ,
3 424 952 631 192 831 306 793 806 306 151 219 954 658 024 452 616 821 761 179 431 217 378 357 803 762 ↘
  908 507 916 210 322 415 778 979 ,
19 962 086 816 615 132 201 030 109 348 061 553 237 699 379 811 537 018 431 693 710 708 365 767 676 708 ↘
  173 510 857 792 660 763 728 460 ,
116 347 568 268 497 961 899 386 849 782 218 099 471 538 254 416 605 288 828 982 833 032 816 248 256 486 ↘
  132 557 230 545 642 166 591 781 ,
678 123 322 794 372 639 195 290 989 345 247 043 591 530 146 688 094 714 542 203 287 488 531 721 862 208 ↘
  621 832 525 481 192 235 822 226 ,
3 952 392 368 497 737 873 272 359 086 289 264 162 077 642 625 711 962 998 424 236 891 898 374 082 916 ↘
  765 598 437 922 341 511 248 341 575 ,
23 036 230 888 192 054 600 438 863 528 390 337 928 874 325 607 583 683 276 003 218 063 901 712 775 638 ↘
  384 968 795 008 567 875 254 227 224 ,
134 264 992 960 654 589 729 360 822 084 052 763 411 168 311 019 790 136 657 595 071 491 511 902 570 913 ↘
  544 214 332 129 065 740 277 021 769 ,
782 553 726 875 735 483 775 726 068 975 926 242 538 135 540 511 157 136 669 567 210 885 169 702 649 842 ↘
  880 317 197 765 826 566 407 903 390 ,
4 561 057 368 293 758 312 924 995 591 771 504 691 817 644 932 047 152 683 359 808 193 819 506 313 328 ↘
  143 737 688 854 465 893 658 170 398 571 ,
26 583 790 482 886 814 393 774 247 481 653 101 908 367 734 051 771 758 963 489 281 952 031 868 177 319 ↘
  019 545 815 929 029 535 382 614 488 036 ,
154 941 685 529 027 128 049 720 489 298 147 106 758 388 759 378 583 401 097 575 883 518 371 702 750 585 ↘
  973 537 206 719 711 318 637 516 529 645 ,
903 066 322 691 275 953 904 548 688 307 229 538 641 964 822 219 728 647 621 966 019 158 198 348 326 196 ↘
  821 677 424 389 238 376 442 484 689 834 ,
5 263 456 250 618 628 595 377 571 640 545 230 125 093 400 173 939 788 484 634 220 231 430 818 387 206 ↘
  594 956 527 339 615 718 940 017 391 609 359 ,
30 677 671 181 020 495 618 360 881 154 964 151 211 918 436 221 419 002 260 183 355 369 426 711 974 913 ↘
  372 917 486 613 305 075 263 661 864 966 320 ,
178 802 570 835 504 345 114 787 715 289 239 677 146 417 217 154 574 225 076 465 911 985 129 453 462 273 ↘
```

```
  642 548 392 340 214 732 641 953 798 188 561 ,
1 042 137 753 832 005 575 070 365 410 580 473 911 666 584 866 706 026 348 198 612 116 541 350 008 798 ∖
  728 482 372 867 427 983 320 588 060 924 165 046 ,
6 074 023 952 156 529 105 307 404 748 193 603 792 853 091 983 081 583 864 115 206 787 262 970 599 330 ∖
  097 251 688 812 227 685 190 886 411 746 801 715 ,
35 402 005 959 107 169 056 774 063 078 581 148 845 451 967 031 783 476 836 492 628 607 036 473 587 181 ∖
  855 027 760 005 938 127 824 730 409 556 645 244 ,
206 338 011 802 486 485 235 336 973 723 293 289 279 858 710 207 619 277 154 840 564 854 955 870 923 761 ∖
  032 914 871 223 401 081 757 496 045 593 069 749 ,
1 202 626 064 855 811 742 355 247 779 261 178 586 833 700 294 213 932 186 092 550 760 522 698 751 955 ∖
  384 342 461 467 334 468 362 720 245 864 001 773 250 ,
7 009 418 377 332 383 968 896 149 701 843 778 231 722 343 055 075 973 839 400 463 998 281 236 640 808 ∖
  545 021 853 932 783 409 094 563 979 138 417 569 751 ,
40 853 884 199 138 492 071 021 650 431 801 490 803 500 358 036 241 910 850 310 233 229 164 721 092 895 ∖
  885 788 662 129 365 986 204 663 628 966 503 645 256 ,
238 113 886 817 498 568 457 233 752 888 965 166 589 279 805 162 375 491 262 460 935 376 707 089 916 566 ∖
  769 710 118 843 412 508 133 417 794 660 604 301 785 ,
1 387 829 436 705 852 918 672 380 866 901 989 508 732 178 472 938 011 036 724 455 379 031 077 818 406 ∖
  504 732 472 050 931 109 062 595 843 138 997 122 165 454 ,
8 088 862 733 417 618 943 577 051 448 522 971 885 803 791 032 465 690 729 084 271 338 809 759 820 522 ∖
  461 625 122 186 743 241 867 441 641 039 322 128 690 939 ,
47 145 346 963 799 860 742 789 927 824 235 841 806 090 567 721 856 133 337 781 172 653 827 481 104 728 ∖
  265 018 261 069 528 342 142 054 003 096 935 649 980 180 ,
274 783 219 049 381 545 513 162 515 496 892 078 950 739 615 298 671 109 297 602 764 584 155 126 807 847 ∖
  128 484 444 230 426 810 984 882 377 542 291 771 190 141 ,
1 601 553 967 332 489 412 336 185 165 157 116 631 898 347 124 070 170 522 447 835 414 851 103 279 742 ∖
  354 505 888 404 313 032 523 767 240 262 156 814 977 160 666 ,
9 334 540 584 945 554 928 503 948 475 445 807 712 439 343 129 122 352 025 389 409 724 522 464 551 646 ∖
  279 906 845 981 647 768 331 618 559 195 398 598 091 773 855 ,
54 405 689 542 340 840 158 687 505 687 517 729 642 737 711 650 663 941 629 888 622 932 283 684 030 135 ∖
  324 935 187 485 573 577 465 944 114 910 234 773 573 482 464 }
```

```
a1[n_Integer] := Expand[(3 + 2 √2)^n]; b1[n_Integer] := Expand[(3 - 2 √2)^n];

aa1[n_] := (a1[n] - b1[n])/(4 √2);

n = 10 000;
Simplify[aa[n]]
Simplify[aa1[n]]
```

$$\frac{-\left(3-2\sqrt{2}\right)^{10\,000}+\left(3+2\sqrt{2}\right)^{10\,000}}{4\sqrt{2}}$$

```
57 694 148 507 840 461 669 872 616 739 192 811 102 482 867 124 113 866 932 060 440 904 737 079 601 384 ∖
  735 500 069 149 112 251 939 919 158 807 922 844 000 632 137 518 388 454 266 875 664 910 140 503 340 510 ∖
  668 691 983 856 215 375 745 600 989 952 798 585 496 506 453 149 741 741 461 272 331 043 381 815 313 258 ∖
  876 614 148 439 428 292 325 842 349 534 401 616 031 933 315 079 126 903 404 365 260 355 581 722 972 094 ∖
  854 494 536 235 510 511 119 347 941 845 811 642 245 298 859 027 028 776 449 973 557 787 259 974 740 589 ∖
  289 337 079 292 500 081 744 495 001 573 928 598 676 002 787 797 707 985 375 670 413 561 771 506 651 248 ∖
  882 614 828 606 657 570 230 876 177 417 029 163 525 722 132 960 450 497 772 475 073 985 092 688 917 781 ∖
  304 575 683 135 513 419 701 537 135 353 263 468 980 433 041 159 746 153 049 083 768 871 911 305 757 614 ∖
  952 089 288 672 283 340 964 626 269 327 198 159 394 296 855 191 899 858 586 867 548 511 955 767 889 434 ∖
  120 373 000 753 702 464 906 558 454 234 593 006 775 251 630 635 847 784 986 670 760 895 892 445 503 128 ∖
  336 526 199 424 731 164 010 871 736 271 794 373 879 135 213 340 320 567 587 840 019 042 406 527 617 816 ∖
  275 874 722 725 699 637 681 173 326 543 332 165 008 542 666 227 661 340 746 749 441 912 566 313 484 371 ∖
  832 776 469 195 139 872 509 768 950 938 392 098 967 615 357 477 954 592 237 007 258 638 755 982 222 934 ∖
```

431 758 271 330 527 576 808 982 717 018 690 723 275 434 718 837 183 066 197 510 901 620 081 644 161 818 ⋱
561 422 329 070 829 974 027 209 860 776 686 643 150 272 262 974 919 614 954 504 919 880 380 225 945 802 ⋱
924 350 065 286 265 738 076 704 588 559 377 501 312 446 847 712 011 337 160 970 162 971 912 684 830 552 ⋱
499 953 523 163 133 668 915 368 953 738 332 615 892 792 122 088 830 303 705 630 201 563 279 389 456 596 ⋱
873 716 176 445 394 372 025 249 122 374 562 541 161 236 528 510 578 637 455 414 377 242 226 130 211 783 ⋱
256 938 346 472 188 747 433 432 072 747 997 159 804 802 963 830 537 366 857 853 527 589 780 721 951 653 ⋱
697 683 131 559 504 829 337 508 610 652 807 053 280 309 996 902 238 299 816 121 317 756 778 364 521 727 ⋱
995 675 433 765 334 165 396 581 170 546 981 845 232 575 360 970 392 295 393 137 957 078 229 632 531 130 ⋱
935 923 613 614 959 409 500 069 721 290 157 905 306 546 991 442 868 897 993 478 030 267 821 380 267 550 ⋱
947 869 057 889 118 147 268 001 010 736 335 762 162 565 926 857 359 913 220 550 366 159 575 671 873 730 ⋱
309 541 595 298 719 470 832 321 342 814 502 815 307 558 357 170 685 666 779 463 755 217 792 065 559 205 ⋱
721 319 210 055 564 815 121 242 809 995 881 551 721 742 885 349 907 940 665 106 992 074 640 333 337 148 ⋱
994 011 298 481 493 599 534 337 647 180 500 553 782 102 175 330 147 212 093 447 948 866 135 959 304 205 ⋱
701 414 990 123 680 717 017 304 774 539 826 730 524 711 713 171 213 010 084 083 118 585 774 375 280 134 ⋱
749 055 087 478 085 023 123 981 579 644 313 023 761 500 234 900 526 076 428 291 951 704 996 868 837 809 ⋱
525 950 268 816 347 198 085 730 139 145 653 280 949 340 894 894 568 974 882 262 410 706 286 709 170 649 ⋱
720 640 845 852 530 390 019 780 124 743 205 827 088 727 600 130 562 770 831 870 187 899 827 822 337 917 ⋱
190 452 009 099 210 172 431 771 309 123 907 855 311 260 786 967 301 933 493 210 671 385 508 468 866 788 ⋱
276 432 507 474 914 475 195 577 782 074 152 814 192 749 590 540 940 086 239 278 494 381 570 354 635 592 ⋱
080 880 868 616 690 236 904 997 460 385 932 279 621 811 249 859 000 338 446 418 562 108 667 283 594 807 ⋱
402 135 194 282 146 698 319 702 671 234 391 730 946 876 963 913 485 397 333 421 511 149 436 581 105 099 ⋱
205 496 106 117 004 905 171 267 355 854 690 773 933 869 302 174 926 036 650 386 734 744 643 589 237 369 ⋱
938 943 992 638 249 352 414 833 532 364 792 689 698 748 050 451 134 726 612 046 725 909 434 570 707 601 ⋱
177 352 244 663 002 356 382 848 224 211 699 466 298 478 436 433 007 405 814 871 693 609 086 739 558 575 ⋱
693 342 230 810 745 156 192 813 462 056 238 054 756 832 426 750 953 461 798 749 196 572 068 406 383 768 ⋱
114 992 998 806 625 431 803 116 607 229 412 669 348 631 326 039 923 798 860 850 770 078 100 368 419 957 ⋱
847 330 382 516 763 884 315 335 852 045 596 722 017 674 648 631 475 297 754 078 320 649 894 600 560 993 ⋱
015 966 181 624 653 178 683 843 325 408 910 598 875 407 333 543 740 940 846 578 986 259 837 534 821 372 ⋱
840 802 872 747 199 041 461 928 296 564 354 509 988 857 551 296 701 075 400 350 562 567 007 926 299 197 ⋱
006 094 878 254 120 775 943 845 811 751 972 130 623 397 660 583 474 600 329 943 550 930 256 717 808 501 ⋱
658 554 659 020 363 062 590 910 210 831 735 542 814 569 127 758 767 740 760 457 910 475 758 308 755 038 ⋱
378 097 666 300 472 926 201 299 107 433 833 144 753 367 529 570 096 388 595 995 738 881 697 779 652 291 ⋱
679 071 654 717 089 953 175 546 667 864 485 193 389 628 547 601 728 224 037 038 496 521 809 858 698 694 ⋱
842 806 859 249 614 435 296 546 071 195 785 851 377 331 108 806 089 189 720 636 615 566 989 681 253 802 ⋱
455 482 901 706 542 153 565 231 496 144 067 012 228 656 523 208 081 432 676 358 072 902 853 592 813 711 ⋱
145 791 810 017 224 675 569 796 981 895 736 658 674 924 651 541 027 327 503 816 125 033 345 668 749 804 ⋱
190 201 548 850 241 868 008 740 385 186 943 950 630 072 964 692 843 116 796 145 192 044 895 303 825 260 ⋱
088 949 542 166 996 475 675 446 758 683 617 598 127 714 153 531 833 930 995 888 365 514 716 630 178 422 ⋱
515 077 551 754 204 155 997 449 836 599 774 974 121 699 475 250 638 792 553 050 465 423 370 364 490 057 ⋱
559 523 013 424 052 496 701 700 710 105 418 610 151 957 026 185 573 682 055 504 374 111 962 021 889 165 ⋱
659 076 327 643 305 437 806 718 180 725 958 006 894 184 463 292 282 258 683 498 953 848 616 343 501 500 ⋱
006 838 436 133 358 400 309 954 031 824 707 720 247 344 877 151 980 705 125 040 152 326 021 421 479 822 ⋱
974 876 710 352 760 571 350 789 000 517 777 914 357 640 377 686 118 586 969 931 483 217 040 740 051 881 ⋱
304 472 358 472 816 620 021 520 583 952 566 640 072 561 916 206 290 736 123 680 997 572 274 344 150 277 ⋱
714 510 491 477 163 995 570 057 082 294 774 897 908 658 783 826 280 223 956 770 000 892 470 480 860 637 ⋱
531 904 712 491 622 088 051 685 338 901 913 221 889 148 464 037 344 116 202 034 649 234 674 389 714 858 ⋱
291 160 935 618 538 514 474 705 773 086 634 775 548 022 423 312 032 700 066 403 523 561 177 321 470 912 ⋱
238 370 088 187 969 392 855 173 942 015 744 644 278 248 544 803 884 367 741 980 299 577 194 943 299 546 ⋱
204 486 039 967 855 238 130 107 357 443 121 880 738 022 748 120 582 057 355 328 282 331 696 034 028 759 ⋱
330 046 561 141 935 172 454 362 956 635 939 580 608 237 537 697 326 796 964 524 787 035 055 025 222 596 ⋱
523 261 464 134 604 738 865 696 825 109 940 690 058 377 688 448 003 028 407 475 187 735 161 704 744 305 ⋱
419 118 130 987 154 643 001 664 730 047 231 065 191 237 786 853 310 414 945 948 069 399 216 280 377 283 ⋱
822 307 317 523 697 072 257 712 623 954 420 534 359 149 419 644 857 695 110 301 720 320 713 738 639 354 ⋱
665 210 649 364 675 262 692 841 531 723 742 864 719 184 894 996 775 950 006 369 681 879 372 188 339 440 ⋱
084 144 556 372 741 311 016 255 019 147 184 964 044 621 427 067 421 162 723 319 318 509 066 882 035 164 ⋱
248 248 692 093 316 476 632 058 572 070 930 017 576 933 411 344 485 950 348 378 345 401 207 451 321 740 ⋱
618 164 478 959 854 679 944 664 110 223 839 174 044 960 373 884 776 432 170 378 881 124 865 293 359 241 ⋱
898 185 533 678 302 429 354 943 641 737 227 869 079 164 317 033 878 589 057 260 786 414 278 629 870 178 ⋱
633 077 517 066 568 229 665 603 872 255 761 929 560 284 047 020 643 291 834 053 252 675 672 183 977 073 ⋱
194 972 139 383 070 957 749 317 880 955 260 507 384 407 979 686 294 287 281 324 659 863 657 838 994 252 ⋱
604 067 938 336 988 574 564 047 318 291 450 690 882 576 547 520 440 987 501 180 468 313 945 071 468 025 ⋱

459 502 183 983 735 019 836 972 571 522 566 429 931 063 981 368 182 133 775 962 400 077 284 771 834 744 ∴
154 499 197 001 446 367 058 591 811 477 434 381 788 302 547 070 377 015 177 185 688 824 243 800 777 175 ∴
918 509 179 216 521 575 528 991 740 255 859 427 940 963 200 008 894 738 267 545 659 567 859 133 030 734 ∴
941 540 771 353 988 930 422 514 561 729 946 767 747 706 208 516 710 767 923 137 673 862 599 394 444 739 ∴
695 963 241 183 630 989 162 458 729 502 353 827 993 314 395 560 142 878 421 596 363 715 330 574 908 601 ∴
159 695 861 691 774 658 319 233 097 705 829 665 515 633 071 879 361 326 946 543 630 571 025 030 657 220 ∴
209 328 991 698 672 927 617 049 515 396 899 553 755 494 930 352 009 411 868 025 197 582 253 462 031 493 ∴
883 991 933 892 313 130 921 020 716 184 751 954 253 721 651 335 051 765 886 247 502 663 351 443 955 852 ∴
637 142 246 023 520 543 742 595 080 591 433 142 345 072 868 138 841 411 352 158 361 688 871 896 756 922 ∴
409 538 378 081 751 791 241 216 054 053 586 872 258 741 464 164 434 705 814 217 249 481 275 478 885 618 ∴
478 290 982 443 035 842 038 590 549 028 821 975 486 755 363 633 705 013 122 735 441 078 601 666 166 415 ∴
861 350 404 097 532 875 340 926 651 164 241 521 028 573 316 534 260 836 408 203 837 658 455 993 988 231 ∴
955 266 086 339 451 790 562 511 160 840 163 960 679 020 301 804 794 960 420 907 709 633 054 294 205 786 ∴
885 353 239 850 278 565 284 083 148 344 520 389 305 959 443 370 827 269 170 271 989 224 029 173 070 644 ∴
038 241 452 992 162 651 228 364 732 543 518 859 194 352 785 465 524 714 593 819 176 180 922 775 101 203 ∴
604 907 066 205 357 491 283 976 699 976 356 014 889 848 497 697 049 612 637 666 628 607 905 692 727 917 ∴
925 534 464 686 621 947 364 700 737 299 038 116 809 596 508 721 131 789 869 165 555 769 966 087 873 804 ∴
773 645 409 870 860 034 782 147 827 497 635 512 999 908 335 587 780 427 398 815 463 006 971 926 710 957 ∴
645 159 097 952 966 615 234 896 045 163 264 004 116 273 903 356 182 956 989 616 006 962 408 181 487 127 ∴
623 377 001 522 250 162 611 020 763 005 505 078 356 114 572 836 335 092 992 438 258 879 291 342 768 146 ∴
926 995 764 245 781 038 657 012 424 278 420 872 341 873 027 151 306 590 392 516 916 385 859 703 813 660 ∴
562 870 953 588 980 004 922 434 729 799 131 544 281 814 740 825 587 559 150 562 880 514 243 298 481 119 ∴
893 039 889 382 232 352 525 409 350 053 296 577 607 281 232 024 534 812 488 166 045 930 424 074 513 575 ∴
296 080 868 534 828 031 779 358 805 681 990 444 198 271 232 362 982 738 946 797 844 278 394 440 345 981 ∴
800 001 954 864

```
n =.; IntegerQ[aa1[n]]
```

```
False
```

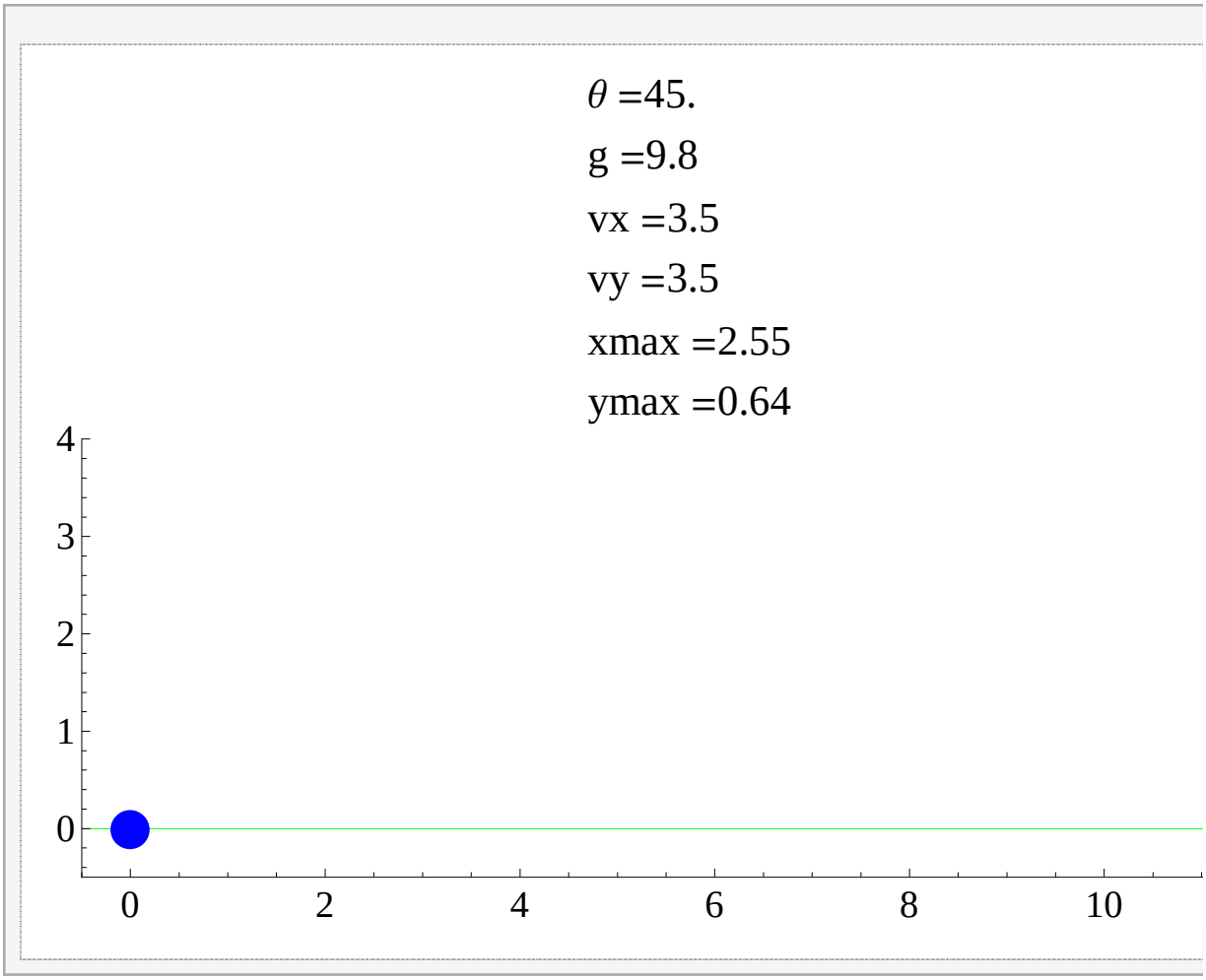
$$\text{IntegerQ}\left[\frac{\sqrt{\text{aa1}[n]+1}-1}{2}\right]$$

```
False
```

Přitažlivost rozhoduje

Dělostřelecké manévry

```
Manipulate[Module[{eqns, soln, x, y, t},
  eqns = {x'[t] == v Cos[Theta], x[0] == 0, y'[t] == -g t + v Sin[Theta], y[0] == 0};
  soln = Flatten[NDSolve[eqns, {x, y}, {t, 0, 2 v Sin[Theta] / g}]];
  (* 2 v Sin[Theta]/g je doba, ked projektil dopadne na zem*)
  With[{d = x[p] /. soln, h = y[p] /. soln},
   Graphics[{{Green, Line[{{-.5, 0}, {12, 0}}]}, {Blue, Disk[{d, h}, .2]}},
    PlotRange → {{-.5, 12}, {-.5, 4}}, Frame → {{True, False}, {True, False}},
    LabelStyle → {Background → White, 18},
    PlotLabel → Style[TableForm[{"θ =" <> ToString[NumberForm[Theta 180 / Pi, {2, 0}]],
        "g =" <> ToString[NumberForm[g, {2, 1}]],
        "vx =" <> ToString[NumberForm[Chop[N[v Cos[Theta]]], {2, 1}]],
        "vy =" <> ToString[NumberForm[Chop[N[v Sin[Theta]]], {2, 1}]],
        "xmax =" <> ToString[NumberForm[N[(v^2 Sin[2 Theta]) / g], {3, 2}]],
        "ymax =" <> ToString[NumberForm[N[(v Sin[Theta])^2 / (2 g)], {3, 2}]]}], 20],
    ImageSize → 600]], {{v, 5, "Počáteční rychlost v"}, 1, 10},
 {{g, 9.8, "Gravitační konstanta"}, 2, 30},
 {{Theta, Pi / 4, "Úhel θ"}, 0, Pi / 2},
 {{p, 0, "Animace"}, 0, 2 v Sin[Theta] / g, ControlType → Trigger},
 ControlPlacement → Right]
```



Mathematica je hra (aneb s hudbou můžeme počítat)

Jaká je to nota

THIS NOTEBOOK IS THE SOURCE CODE FROM

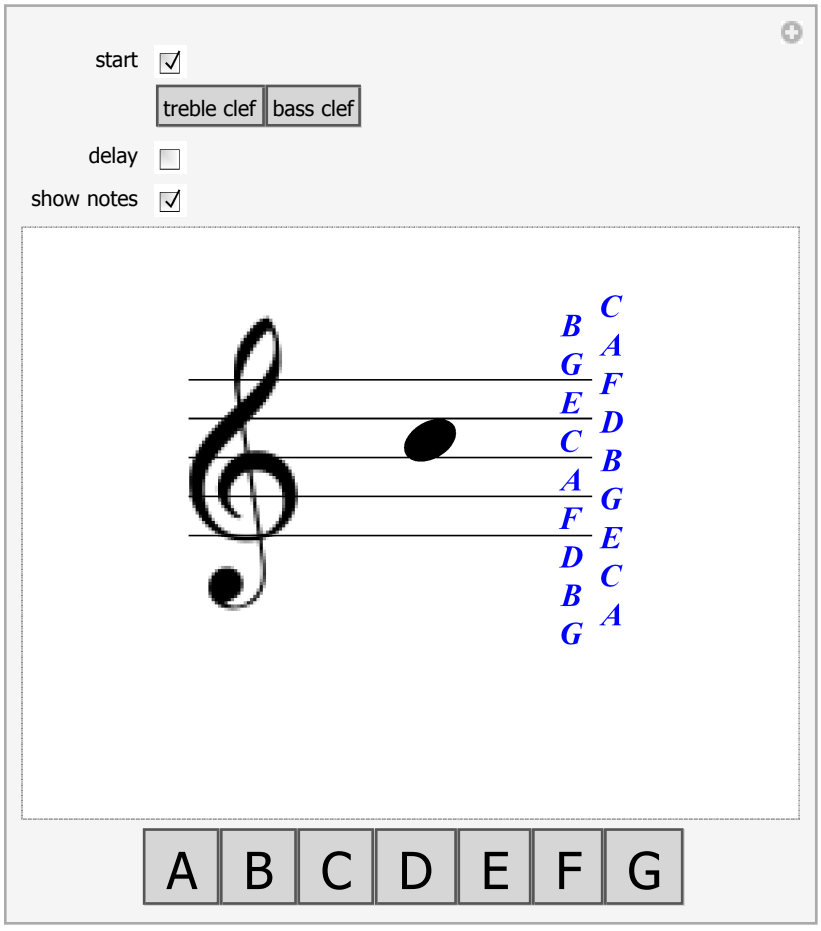
"Learn Musical Notes" from The Wolfram Demonstrations Project
http://demonstrations.wolfram.com/LearnMusicalNotes/

```
Manipulate[
```

```
trebleclef = 𝄞;
```

```
bassclef = 𝄢;

scale = {"A", "B", "C", "D", "E", "F", "G"};

(* determine either to use clef notes or bass notes on the side as a guide*)
clef2 = If[clef === trebleclef, Drop[RotateLeft[Join[scale, scale, scale], 6], -3],
  Drop[RotateLeft[Join[scale, scale, scale], 1], -3]];

(* checks selected answer *)
If[checkanswer[choices, notenum, scale, clef],
 notenum = RandomChoice[Range[-.5, 8, .5]]; choices = 0;, notenum];

(* generates note based on location on the staff *)
createnote[location_] :=
 Which[location ≥ 7, Graphics[Flatten@{noteshape, Line[{{-1, #}, {1, #}}] & /@
      Range[46 - Abs[location - 7], 46 - If[Mod[location, 1] == 0, 0, .5]]},
   ImageSize → {24., Automatic}], 1 < location < 7, Graphics[noteshape,
   ImageSize → {17., Automatic}], location ≤ 1, Graphics[Flatten@
    {noteshape, Line[{{-1, #}, {1, #}}] & /@ Range[46 + If[Mod[location, 1] == 0, 0, .5],
       46 + Abs[location - 1]]}, ImageSize → {24., Automatic}]];

(* generates sound *)
createsound[x_, note_, sc_] :=
 Module[{answer, sc2}, answer = (If[clef === trebleclef, sc2 = sc, sc2 = RotateLeft[sc, 2]];
    sc2[[If[Mod[Mod[note, 7] + Mod[note, 7] + 1, 7] == 0,
      7, Mod[Mod[note, 7] + Mod[note, 7] + 1, 7]]]]);
  EmitSound[Sound[SoundNote[answer <> ToString[x]]]]];

(* check selected answer and display result *)
checkanswer[guess_, note_, sc_, cl_] := Module[{answer, sc2},
  answer = (If[cl === trebleclef, sc2 = sc, sc2 = RotateLeft[sc, 2]];
    sc2[[If[Mod[Mod[note, 7] + Mod[note, 7] + 1, 7] == 0,
      7, Mod[Mod[note, 7] + Mod[note, 7] + 1, 7]]]]);
  If[guess == answer, displayedtext2 = ""; If[cl === bassclef,
    Which[note < 0, createsound[1, note, sc], 0 <= note < 3.5, createsound[2, note, sc],
     3.5 <= note < 7., createsound[3, note, sc], note >= 7., createsound[4, note, sc]],
    Which[note <= .5, createsound[3, note, sc], .5 < note ≤ 4., createsound[4, note, sc],
     4 < note <= 7.5, createsound[5, note, sc], note > 7.5, createsound[6, note, sc]]];
   If[delay, Pause[1], Pause[0]];, If[guess === 0, displayedtext2 = "",
    displayedtext2 = Style["Try Again", Red];]]; guess === answer];

(* Determine which note graphic to draw based on notenum *)
Deploy@Magnify[Grid[{{Graphics[{
       Table[Line[{{0, y}, {200, y}}], {y, 0, offset, offset / 4.}],
       If[start == True, Inset[createnote[notenum], {120, Which[notenum == 8, 122,
           1 <= notenum <= 7.5, offset / 4. * notenum - offset / 4. * 2.02, notenum == -.5,
           -42, notenum == 0, -38, notenum == .5, -33]}], Inset[""]], Inset[clef, {32, 43}],
       If[togglenotes, Table[Inset[Style[clef2[[i]], Blue, Bold, Italic, 10],
          {If[EvenQ[i], 210, 190], (i - 6.1) offset2}], {i, 1, 18}],
        Inset[""]], PlotRange -> {{0., 138}, {100, 147.}}
      }, AspectRatio -> 1.02`,
ImageSize -> {220., 141.},
PlotRange -> {{0., 220.}, {-100, 147.}}
     ]}, {Grid[{{Dynamic[displayedtext2]}}, BaseStyle → "Subsection"]}}], 1.5],
{{start, False}, {False, True}}, {{clef, trebleclef, ""},
 {trebleclef → "treble clef", bassclef → "bass clef"}, ControlType → SetterBar},
```

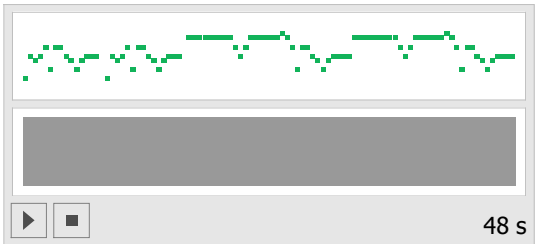
```
{{delay, True, "delay"}, {True, False}},
{{togglenotes, False, "show notes"}, {True, False}},
{{notenum, RandomChoice[Range[-.5, 8, .5]]}, ControlType → None},
{{clef2, trebleclef}, ControlType → None},
{{displayedtext2, ""}, ControlType → None},
{{choices, 0, "                "},
 (# → Style[" " <> # <> " ", 24]) & /@ {"A", "B", "C", "D", "E", "F", "G"},
 ControlType → SetterBar, ControlPlacement → Bottom},
TrackedSymbols :→ {clef, start, togglenotes, notenum, choices},
SynchronousUpdating → False, AutorunSequencing → {2, 4},
Initialization :→ {offset = 85, offset2 = 10.5,
  noteshape = Rotate[Disk[{0, 46}, {0.76, 0.5}], Pi / 6.];}
```



Elvis žije

"Accordion"	"Agogo"	"AltoSax"	"Applause"
"Atmosphere"	"Bagpipe"	"Bandoneon"	"Banjo"
"BaritoneSax"	"Bass"	"BassAndLead"	"Bassoon"
"Bird"	"BlownBottle"	"Bowed"	"BrassSection"
"Breath"	"Brightness"	"BrightPiano"	"Calliope"
"Celesta"	"Cello"	"Charang"	"Chiff"
"Choir"	"Clarinet"	"Clavi"	"Contrabass"
"Crystal"	"DrawbarOrgan"	"Dulcimer"	"Echoes"
"ElectricBass"	"ElectricGrandPiano"	"ElectricGuitar"	"ElectricPiano"
"ElectricPiano2"	"EnglishHorn"	"Fiddle"	"Fifths"
"Flute"	"FrenchHorn"	"FretlessBass"	"FretNoise"
"Glockenspiel"	"Goblins"	"Guitar"	"GuitarDistorted"
"GuitarHarmonics"	"GuitarMuted"	"GuitarOverdriven"	"Gunshot"
"Halo"	"Harmonica"	"Harp"	"Harpsichord"
"Helicopter"	"HonkyTonkPiano"	"JazzGuitar"	"Kalimba"
"Koto"	"Marimba"	"MelodicTom"	"Metallic"
"MusicBox"	"MutedTrumpet"	"NewAge"	"Oboe"
"Ocarina"	"OrchestraHit"	"Organ"	"PanFlute"
"PercussiveOrgan"	"Piano"	"Piccolo"	"PickedBass"
"PizzicatoStrings"	"Polysynth"	"Rain"	"Recorder"
"ReedOrgan"	"ReverseCymbal"	"RockOrgan"	"Sawtooth"
"SciFi"	"Seashore"	"Shakuhachi"	"Shamisen"
"Shanai"	"Sitar"	"SlapBass"	"SlapBass2"
"SopranoSax"	"Soundtrack"	"Square"	"Steeldrums"
"SteelGuitar"	"Strings"	"Strings2"	"Sweep"
"SynthBass"	"SynthBass2"	"SynthBrass"	"SynthBrass2"
"SynthDrum"	"SynthStrings"	"SynthStrings2"	"SynthVoice"
"Taiko"	"Telephone"	"TenorSax"	"Timpani"
"Tinklebell"	"TremoloStrings"	"Trombone"	"Trumpet"
"Tuba"	"TubularBells"	"Vibraphone"	"Viola"
"Violin"	"Voice"	"VoiceAahs"	"VoiceOohs"
"Warm"	"Whistle"	"Woodblock"	"Xylophone"

```
Sound[{"Guitar", SoundNote["D", .5], SoundNote["G", .5], SoundNote["FSharp", .5],
  SoundNote["G", .5], SoundNote["A", .5], SoundNote["E", .5], SoundNote["A", 1],
  SoundNote["G", .5], SoundNote["FSharp", .5], SoundNote["E", .5],
  SoundNote["FSharp", .5], SoundNote["G", 1.5], SoundNote[None, .5],
  SoundNote["D", .5], SoundNote["G", .5], SoundNote["FSharp", .5], SoundNote["G", .5],
  SoundNote["A", .5], SoundNote["E", .5], SoundNote["A", 1], SoundNote["G", .5],
  SoundNote["FSharp", .5], SoundNote["E", .5], SoundNote["FSharp", .5],
  SoundNote["G", 1.5], SoundNote[None, .5], SoundNote["B", .5], SoundNote["B", .5],
  SoundNote["B", .5], SoundNote["B", .5], SoundNote["B", .5], SoundNote["B", .5],
  SoundNote["B", 1], SoundNote["B", .5], SoundNote["A", .5], SoundNote["G", .5],
  SoundNote["A", .5], SoundNote["B", 2], SoundNote["B", .5], SoundNote["B", .5],
  SoundNote["C5", .5], SoundNote["B", .5], SoundNote["A", .5], SoundNote["E", .5],
  SoundNote["A", 1], SoundNote["G", .5], SoundNote["FSharp", .5],
  SoundNote["E", .5], SoundNote["FSharp", .5], SoundNote["G", 2], SoundNote["B", .5],
  SoundNote["B", .5], SoundNote["B", .5], SoundNote["B", .5], SoundNote["B", .5],
  SoundNote["B", .5], SoundNote["B", 1], SoundNote["B", .5], SoundNote["A", .5],
  SoundNote["G", .5], SoundNote["A", .5], SoundNote["B", 2], SoundNote["B", .5],
  SoundNote["B", .5], SoundNote["C5", .5], SoundNote["B", .5], SoundNote["A", .5],
  SoundNote["E", .5], SoundNote["A", 1], SoundNote["G", .5], SoundNote["FSharp", .5],
  SoundNote["E", .5], SoundNote["FSharp", .5], SoundNote["G", 2]}]
```



Dzivočka v kostele

```
Sound[{"Organ", SoundNote["G", .5], SoundNote["G", .5], SoundNote["FSharp", .5],
  SoundNote["E", .5], SoundNote["B3", .5], SoundNote["A", .5], SoundNote["G", .5],
  SoundNote[None, .5], SoundNote["FSharp", .25], SoundNote["E", .25],
  SoundNote["DSharp", .25], SoundNote["FSharp", .25], SoundNote["E", .5],
  SoundNote[None, .5], SoundNote["G", .5], SoundNote["G", .5],
  SoundNote["FSharp", .5], SoundNote["E", .5], SoundNote["B3", .5],
  SoundNote["A", .5], SoundNote["G", .5], SoundNote[None, .5],
  SoundNote["FSharp", .25], SoundNote["E", .25], SoundNote["DSharp", .25],
  SoundNote["FSharp", .25], SoundNote["E", .5], SoundNote[None, .5],
  SoundNote["G", .5], SoundNote["B", .5], SoundNote["B", .5], SoundNote["B", .5],
  SoundNote["A", .5], SoundNote["C5", .5], SoundNote["B", .5], SoundNote[None, .5],
  SoundNote["A", .25], SoundNote["G", .25], SoundNote["FSharp", .25],
  SoundNote["G", .25], SoundNote["A", .5], SoundNote["ASharp", .5],
  SoundNote["B", .5], SoundNote[None, .5], SoundNote["G", .5],
  SoundNote["G", .5], SoundNote["FSharp", .5], SoundNote["E", .5],
  SoundNote["B3", .5], SoundNote["A", .5], SoundNote["G", .5], SoundNote[None, .5],
  SoundNote["FSharp", .25], SoundNote["E", .25], SoundNote["DSharp", .25],
  SoundNote["FSharp", .25], SoundNote["E", .5], SoundNote[None, .5],
  SoundNote["G", .5], SoundNote["B", .5], SoundNote["B", .5], SoundNote["B", .5],
  SoundNote["A", .5], SoundNote["C5", .5], SoundNote["B", .5], SoundNote[None, .5],
  SoundNote["A", .25], SoundNote["G", .25], SoundNote["FSharp", .25],
  SoundNote["G", .25], SoundNote["A", .5], SoundNote["ASharp", .5],
  SoundNote["B", .5], SoundNote[None, .5], SoundNote["G", .5],
  SoundNote["G", .5], SoundNote["FSharp", .5], SoundNote["E", .5],
  SoundNote["B3", .5], SoundNote["A", .5], SoundNote["G", .5], SoundNote[None, .5],
  SoundNote["FSharp", .25], SoundNote["E", .25], SoundNote["DSharp", .25],
  SoundNote["FSharp", .25], SoundNote["E", .5], SoundNote[None, .5]}]
```